MODELLING INFECTIOLOGY AND OPTIMAL CONTROL OF DENGUE FEVER DISEASE EPIDEMICS IN TANZANIA

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CERTIFICATION

The undersigned certifies that they have read and hereby recommends for acceptance by the Open University of Tanzania a thesis entitled: “**Modelling Infectiology and Optimal Control of Dengue Fever Disease Epidemics in Tanzania**” in fulfillment of requirements for the award of the degree of Doctor of Philosophy of the Open University of Tanzania

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DECLARATION

I, **Laurencia Ndelamo Massawe**, do hereby declare that this thesis substantially is my own original work and that it has not been submitted and will not be submitted to any other University for a similar or any other degree award.

Signature……………………

Date ………………………...

DEDICATION

This Thesis is dedicated to my family, my friends who encourage and support me and who ever have been with me both physically and spiritually.

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Harvesting time is always more joyous than sowing. These in most cases make us forget those who might have helped us before. It will be unfair and forgetfulness to count this success on my effort only. This work is the product of collective effort!

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ABSTRACT

A mathematical model for infectiology and optimal control of dengue fever disease epidemics in Tanzania is formulated and analysed. The model describes the interaction between human and dengue fever mosquito populations with treatment. Susceptible human population is divided into two, namely, careful and careless susceptible. The model presents two disease-free and two endemic equilibrium points. The results show that the disease-free equilibrium point is locally and globally asymptotically stable if the reproduction number is less than unity. Endemic equilibrium point is locally and globally asymptotically stable under certain conditions using additive compound matrix and Lyapunov method respectively.

The model is fitted to data on dengue fever disease using maximum likelihood estimator. From the results, it is observed that the forecasted data closely agree to the actual data. Sensitivity analysis of the model is implemented in order to investigate the sensitivity of certain key parameters of dengue fever disease transmission.

Moreover the model consists of five control strategies that is campaign aimed in educating careless individuals, reducing mosquito-human contact, removing vector breeding places, insecticide application and the control effort aimed at reducing the maturation rate from larvae to adult. Optimal Control (OC) approach is used in order to find the best strategy to fight the disease and minimize the cost. From the cost-effectiveness analysis, the results suggest that combination of removing vector breeding places and reducing maturation rate from larvae to adult is the most cost-effective of all the strategies for dengue fever disease control considered.

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CHAPTER ONE

INTRODUCTION

1.1 General Introduction

Dengue is a major health problem found in tropical and sub-tropical climates worldwide, mostly in urban and semi-urban areas [http://www.who. Int/mediacentre /factsheets/fs117/en/]. Dengue fever disease can cause a severe flu-like illness, and sometimes Dengue fever can vary from mild to severe. The more severe forms of dengue fever include dengue hemorrhagic fever and dengue shock syndrome. Dengue fever (DF) is a vector-borne disease transmitted by female Aedes aegypti and Aedes albopictus mosquitoes because they require blood meal for the development of their eggs. Four different serotypes can cause dengue fever. A human infected by one serotype, on recovery, gains total immunity to that serotype and only partial and transient immunity with respect to the other three. Preventing or reducing dengue virus transmission depends entirely on the control of mosquito. The spread of dengue is attributed to expanding geographic distribution of the four dengue viruses and their mosquito vectors (Rodrigues *et al*., 2010).

Dengue disease in more severe cases is associated with loss of appetite, vomiting, high fever, headache, abdominal pain, shock and circulatory failure. Dengue remains a serious threat for human health in Tanzania because an effective dengue vaccine and anti-viral treatment are not currently available (Rodrigues *et al*., 2013, Massawe *et al*., 2015).

The presence of Aedes aegypti mosquito was first identified in Dares Salaam city in Tanzania followed by few regions in the year 2014. In July 2010 for the first time in

Tanzania, an outbreak of dengue fever was reported and over 40 people were infected. From 2010 to 2015 the number of infected cases has been increasing.

Moreover in the year 2014 the government of Tanzania announced the danger of the disease where people were alerted about the disease and the precaution to be taken, symptoms, prevention and lifesaving like getting plenty of bed rest, drinking lots of fluids, taking medicine to reduce fever, taking pain relievers with acetaminophen and avoid those containing aspirin, and if is most severe form that is Dengue hemorrhagic fever and Dengue shock syndrome, early and aggressive emergency treatment can be lifesaving by:

i) Emergency treatment with fluid and electrolyte replacement,
ii) Blood pressure monitoring,
iii) Transfusion to reduce blood loss,

Management of severe form of dengue hemorrhagic fever frequently requires hospitalization, for example treating electrolyte imbalances caused by kidney failure can be difficult, because many medicines lower some electrolyte levels while raising other levels. Doctor needs regularly monitor electrolyte levels.

In the year 2014 between January and December and January to April 2015, 1025 people were infected with dengue fever disease and 4 died of the disease from Dar es Salaam city. The regions which were affected with Dengue fever disease are Dar es Salaam with 1014 cases (Kinondoni-601, Temeke-144, and Ilala-269), Kigoma3, Mwanza2, Mbeya2, Kilimanjaro 3 and Njombe1 (ministry of health and social welfare in Tanzania, Massawe *et al.*, 2015).

Currently 2.5 billion people living in areas at risk of DF transmission, each year, an estimated 100 million cases of dengue fever occur worldwide (Gibbons and Vaughn, 2002; WHO, 2002). The disease create many burdens on families as some bread winners and also the governments which have to spent millions of money in diagnosing the disease, purchasing of pesticides to kill the mosquitoes and purchasing drugs to treat the patients and the other intervention schemes. It is these effects of the disease that call for continuous research into the prevention and control of the disease by using Optimal Control theory.

Optimal Control theory is a powerful Mathematical tool used to make decisions on how to control epidemiologic diseases like dengue fever disease. Optimal Control theory is used to minimize the investments in disease's control, since the financial resources are always scarce. Quantitative methods are applied to the optimization of investments in the control of the epidemiologic disease, in order to obtain a maximum of a benefit from a fixed amount of financial resources (Rodrigues *et al.*, 2010). Optimal control theory helps to find the percentage of the individuals who should be treated as time evolves in a given epidemic model in order to minimize the spread of disease and the cost of implementing the treatment strategy (Lenhart & Workman, 2007). In a dynamical system, the optimal control problem for ordinary differential equations is described by the state equation: $g_i^{'}(t) = q(t,\ g_i, u(t))$ where

$u(t)$ is control and g_i is the state variables which depend on the control variables. The control enters the system of differential equations and adjusts the dynamics of the state system. The goal is to adjust the control in order to maximize (or minimize) a given objective function subject to some constraints (Lenhart & Workman, 2007). The aim of the control is to minimize the objective function

i.e. $J = \min_{u} \int_{t_0}^{t} \left(t, g_i, u(t)\right) dt$ (1.1)

subject to the differential equations and initial conditions. Such a minimizing control is called an optimal control problem (Lenhart & Workman, 2007).

The principle technique for such an optimal control problem is to solve a set of "necessary conditions" that an optimal control and corresponding state variables must satisfy. The necessary conditions is generated from the Hamiltonian H, is defined as $H(t, g_i, u, \lambda) = f(t,\ \mathrm{g}_i,\ u) + \lambda q(t,\ \mathrm{g}_i,\ u)$ (1.2)

$\Rightarrow H = \text{integrand of } (1.1) + \text{ adjoint} \times \text{Right hand side (RHS) of } (1.2)$

Then it is intended to minimize H with respect to $u = u^*$ (optimal control) and the conditions is written in terms of the Hamiltonian:

$\frac{\partial H}{\partial u} = 0 \text{ at } u^* \Rightarrow f_u + \lambda q_u = 0$ (Optimality condition),
$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial g_i} \Rightarrow \frac{d\lambda}{dt} = -\left(f_{g_i} + \lambda g_i\right)$ (Adjoint equation),
$\lambda(t_1) = 0$ (Transversality condition),

The dynamics of the state equation is given by
$g_i^{'} = q(t, g_i, u) = \frac{\partial H}{\partial \lambda_i},\ \ \mathrm{g}(t_0) = g_0$ (Lenhart & Workman, 2007)

Mathematical models have played a major role in increasing our understanding of the dynamics of infectious diseases. Several models have been proposed to study the effects of some factors on the transmission dynamics of these infectious diseases including Dengue fever and to provide guidelines as to how the spread can be controlled (Seidu and Makinde, 2014).

Mathematical modelling also became considerable important tool in the study of epidemiology because it helps us to understand the observed epidemiological patterns, disease control and provide understanding of the underlying mechanisms which influence the spread of disease and may suggest control strategies (Ozaire *et al*., 2012). The epidemiological data and the economic cost of infectious diseases are effective elements in evaluating the relevance of intervention programmes. In economic situation, any intervention, like treatment, that has been found to be cost effective would be fully funded without delay. Sometimes, funding and access to treatment may be difficult as always faced with a number of constraints. Optimal control theory to determine the optimal resource allocation as an epidemic progress has been used. Optimal control theory is a powerful mathematical tool to make decision involving complex dynamical systems (Lenhart & Workman, 2007). For example, what percentage of the population should be treated as time evolves in a given epidemic model to minimize both the number of infected people and the cost of implementing the treatment strategies. Optimal control problems have generated a lot of interest from researchers all over the world, for instance in Thome *et al*., (2010), the authors presented a mathematical model of optimal control by considering the cost of insecticide application, the cost of the production of irradiated mosquitoes and their delivery as well as the social cost. In Rodrigues *et al*., (2012), the authors used three vector control tools: larvicide, adulticide and mechanical control. In Rodrigues *et al.*, (2010), the authors presented an application of optimal control theory to Dengue epidemics. The dynamic model is described by a set of nonlinear ordinary differential equations that depend on the dynamics of the Dengue mosquito, the number of infected individuals, and the people's motivation to combat the mosquito. The cost functional depends not only on the costs of medical treatment of the infected people but also on the costs related to educational and sanitary campaigns. They used two approaches to solve the problem: one using optimal control theory, another one by discretizing first the problem and then solving it with nonlinear programming, leading to a decrease of infected mosquitoes and individuals in less time and with lower costs. In Rodrigue *et al*., (2011), the authors used optimal control theory for control of the vector that is mosquito. Their model consists of eight mutually exclusive compartments representing the human and vector dynamics. It also includes a control parameter (insecticide) in order to fight the mosquito. In Ozair *et al*., (2012), the authors used the optimal control theory in which their model consists of three control measures; the preventive control to minimize vector-human contacts, the treatment control to the infected human, and the insecticide control to the vector.

Not much research has been done in the study of epidemic models that consider all five controls, campaign aimed in educating careless individuals, reducing mosquito-human contact, removing vector breeding places, insecticide application, reducing the maturation rate from larvae to adult in Tanzania.

Moreover in Massawe *et al.*, (2015), the authors presented a dynamical model that studied the temporal model for dengue disease with treatment. In this work, the model by Massawe *et al.*, (2015) will be extended, to include rate at which recovery individual lose immunity, Positive change in behaviour for carless susceptible, average daily biting rate per day for careful susceptible, average daily biting rate per day for careless susceptible, average daily biting rate per day for mosquito and Susceptibles with different behaviour that is the dynamical system that incorporates the effects of Careful and Careless human susceptible on the transmission of Dengue fever disease, and then the use of all five controls, that is campaign aimed in educating careless individuals, reducing mosquito-human contact, removing vector breeding places, insecticide application, reducing the maturation rate from larvae to adult.

The model to include various intervention strategies to obtain an optimal control problem will be analysed qualitatively using the Pontryagin's Maximum principle. The resulting optimal control problem is also solved numerically to gain more insights into the implications of the interventions.

1.2 Statement of the Problem:

The study is motivated by the fact that the disease is endemic and claims many lives. Dengue fever (DF) is still endemic in many countries, in those with tropical and sub-tropical climates, including Tanzania. Areas of on-going transmission are as shown below [www.healthmap.org/dengue/index.php].

Figure 1. 1: Map showing global distribution of Dengue fever

Figure 1. 2: Map showing distribution of dengue in Africa and the Middle East

Mathematical models of the dynamics of this disease with special emphasis on Tanzania are uncommon. Few studies of optimal control has been carried out such as those by Thome *et al*., (2010), Rodrigues *et al*., (2011),Wijayaa *et al*., (2013)**,** Ozair *et al*., (2012), which have applied optimal control theory. In particular Massawe *et al*., (2015), studied a temporal model for dengue disease with treatment.

However, none of these studies have considered the rate at which recovery individual lose immunity, positive change in behaviour for carless susceptible, average daily biting rate per day for careful susceptible, average daily biting rate per day for careless susceptible, average daily biting rate per day for mosquito, susceptible with different behaviour that is the dynamical system that incorporates the effects of careful and careless human susceptible on the transmission of Dengue fever disease, and then the use of all five controls, that is campaign aimed in educating careless individuals, reducing mosquito-human contact, removing vector breeding places, insecticide application, reducing the maturation rate from larvae to adult in Tanzania. Therefore, this study intends to apply optimal control approach using Pontryagin's Maximum principle in order to find the best strategy to fight the disease and minimize the cost.

1.3 Research Objectives

The main objective of this study is to formulate and analyse a mathematical model for Infectiology and Optimal Control of Dengue Fever Disease.

The specific objectives are:

(i) To formulate a mathematical model for Infectiology and Optimal Control of Dengue Fever Disease.

(ii) To determine the existence and stabilities of the disease free equilibrium point and endemic equilibrium point.

(iii) To perform sensitivity analysis of mathematical models on the effect of each parameter on the spread or control of Dengue fever disease.

(iv) To determine the impact of optimal control strategies on the spread of Dengue fever disease in the perspective of health care and the society.

(v) To determine the impact of each embedded parameter on the model.

1.4 Research Hypotheses

(i) It is possible to formulate a mathematical model of optimal control of Dengue fever disease for minimizing the spread of Dengue fever disease and minimizing the cost involved in the control strategies.

(ii) The disease free equilibrium point is stable if the reproduction number is less than unity and endemic equilibrium point is stable if the reproduction number is greater than unity.

(iii) Sensitivity analysis will reveal the most sensitive parameter on the spread of Dengue fever disease.

(iv) Optimal control of Dengue fever disease can make the control strategies affordable to the health care and the society.

(v) The variation of the embedded parameters and the results of the model of optimal control of Dengue fever disease are positively correlated.

1.5 Significance of the Study

The health as well as the socioeconomic impact of emerging and re-emerging infectious diseases is significant. This study is significant for the following reasons:

The study will help health care sectors to optimize controls and minimizing the cost of control strategies in order to reduce the spread of Dengue fever disease in the society. It will also help to improve control strategies for the occurrence of Dengue fever disease outbreak in the community of Tanzania. Through this study, public health policy makers will be guided on optimal control strategies which they can consider to control the disease. The study will be used to inform national health authorities about the burden of Dengue fever disease and the economic value of implementing the campaigns of making the society to be careful in Dengue fever transmission in Tanzania. Furthermore, the study will help educationalist to develop educational seminars, workshops or training programmes to educate people about the control strategies of Dengue fever disease. The study will also act as a platform for further research on optimal control of Dengue fever disease and form a base for further studies of related problem.

CHAPTER TWO

LITERATURE REVIEW

Dengue has become a serious health problem worldwide and researchers have focused their attention on understanding how the dengue fever disease is treated. A number of studies have been conducted to highlight the control of the disease.

Massawe *et al*., (2015) presented a mathematical model for the dengue fever disease with treatment. Comprehensive mathematical techniques were used to analyse stability of the model. It was found that the disease free equilibrium point is locally and globally asymptotically stable if the reproduction number (R_0) is less than unity. Then the dengue fever model's endemic is locally asymptotically stable when reproduction number is greater than unity. Sensitivity indices of R_0 to the parameters in the model were calculated. The sensitivity indices revealed that the average daily biting, maturation rate from larvae to adult, transmission probability from human to mosquito, number of larvae per human, transmission probability from mosquito to human and the number of eggs at each deposit per capita, when each one increases keeping the other parameters constant they increase the value of R_0 implying that they increase the endemicity of the disease. While other parameters, average lifespan of humans, natural mortality of larvae, mean viremic period and average lifespan of adult mosquitoes, decrease the value of R_0 implying that they decrease the endemicity of the disease. The numerical simulations were performed using a set of reasonable parameter values. The results suggest that treatment have a positive impact on the decrease of growth rate of dengue fever disease, and the number of death is reduced.

Evans *et al.,* (2014) presented a simple mathematical model to replicate the key features of the sterile insect technique (SIT) for controlling pest species, with particular reference to the mosquito Aedes aegypti, the main vector of dengue fever. The spatial uniform equilibria of the model were identified and analysed. Simulations were performed to analyse the impact of varying the number of release sites, the interval between pulsed releases and the overall volume of sterile insect releases on the effectiveness of SIT programmes.

Results show that, given a fixed volume of available sterile insects, increasing the number of release sites and the frequency of releases increase the effectiveness of SIT programmes. It was also observed that programmes may become completely ineffective if the interval between pulsed releases is greater that a certain threshold value and that, beyond a certain point, increasing the overall volume of sterile insects released does not improve the effectiveness of SIT. It was also noted that insect dispersal drives a rapid recolonisation of areas in which the species has been eradicated and they argued that understanding the density dependent mortality of released insects was necessary to develop efficient, cost-effective SIT programmes.

Aldila *et al.,* (2013) presented an optimal control problem for a host-vector Dengue transmission model. In the model, treatments with mosquito repellent were given to adults and children and those who undergo treatment were classified in treated compartments. With this classification, the model consists of 11 dynamic equations. The basic reproductive ratio that represents the epidemic indicator was obtained from the largest eigenvalue of the next generation matrix. The optimal control problem was designed with four control parameters, namely the treatment rates for children and adult compartments, and the drop-out rates from both compartments. The cost functional accounts for the total number of the infected persons, the cost of the treatment, and the cost related to reducing the drop-out rates. Numerical results for the optimal controls and the related dynamics were shown for the case of epidemic prevention and outbreak reduction strategies. The significance of the age structure was indicated in the calculation of the optimal cost. The higher cost value in the case with no age structure is simply due to the use of adults unit treatment cost for all persons. With a limited budget, it is much better to apply the treatment well before the occurrence of the outbreak.

Thome *et al.*, (2010) presented a mathematical model to describe the dynamics of mosquito population when sterile male mosquitoes (produced by irradiation) were introduced as a biological control, besides the application of insecticide. The effort was made to reduce the fertile female mosquitoes, by searching for the optimal control considering the cost of insecticide application, the cost of the production of irradiated mosquitoes and their delivery as well as the social cost (proportional to the number of fertilized female’s mosquitoes). The optimal control is obtained by applying the Pontryagin’s Maximum Principle (The powerful method for computation of optimal controls).

The results were that high application of insecticide is needed at the beginning of the control, with an exponential decay. Furthermore, the release of insects in general follows a bell shape distribution with an abrupt increasing and decreasing at the extremes, and a plateau at the middle, except in the case when social cost is increasing one hundred times.

Rodrigues *et al*., (2012) developed a model with six mutually-exclusive compartments related to Dengue disease. In their model, there are three vector control tools: larvicide, adulticide and mechanical control. The problem was studied using an Optimal Control (OC) approach. Simulations based on clean-up campaigns to remove the vector breeding sites, and also simulations on the application of insecticides (larvicide and adulticide), were made. It was shown that even with a low, although continuous, index of control over the time, the results were surprisingly positive. The adulticide was the most effective control, from the fact that with a low percentage of insecticide, the basic reproduction number is kept below unity and the infected number of humans was smaller.

Rodrigues *et al*., (2010) presented a model for the transmission of dengue disease. It consists of eight mutually-exclusive compartments representing the human and vector dynamics. It also includes a control parameter (insecticide) in order to fight the mosquitoes. The main goal of this work was to investigate the best way to apply the control in order to effectively reduce the number of infected human and mosquitoes. The numerical tests conclude that the best strategy for the infected reduction was the weekly administration although it was the most expensive one (insecticide cost). The best result obtained was between 11 and 12 days, with the insecticide amount in the closed interval from 7 to 8, confirming the amount of constant control strategy. The 11 or 12 days between applications can be directly related to the span of adult stage for the mosquitoes, an average of eleven days in an urban environment.

Rodrigues *et al*., (2010) presented an application of optimal control theory to Dengue epidemics. The dynamic model was described by a set of nonlinear ordinary differential equations that depend on the dynamic of the Dengue mosquito, the number of infected individuals, and the people's motivation to combat the mosquito. The cost functional depends not only on the costs of medical treatment of the infected people but also on the costs related to educational and sanitary campaigns. Two approaches to solve the problem were considered: one using optimal control theory, another one by discretizing first the problem and then solving it with nonlinear programming. They observed that after four weeks the percentage of infected mosquitoes vanishes and the number of infected individuals begin to decrease, leading to much smaller cost with insecticides and educational campaigns.

Ozair *et al*., (2012) presented a model for the transmission dynamics of a vector-borne disease with nonlinear incidence rate. It was proved that the global dynamics of the disease were completely determined by the basic reproduction number. In order to assess the effectiveness of disease control measures, the sensitivity analysis of the basic reproductive number R_0 and the endemic proportions with respect to epidemiological and demographic parameters were provided. From the results of the sensitivity analysis, the model was modified to assess the impact of three control measures; the preventive control to minimize vector-human contacts, the treatment control to the infected human, and the insecticide control to the vector. Analytically the existence of the optimal control was established by the use of an optimal control technique and numerically it was solved by an iterative method. Numerical simulations and optimal analysis of the model show that restricted and proper use of control measures might considerably decrease the number of infected humans in a viable way. It was found from the sensitivity indices analysis that the most sensitive parameters were those of mosquito biting and death rates. The work was also extended to assess the impact of some control measures. By the application of optimal control theory, they derived and analysed the conditions for optimal control of the disease with personal protection, treatment and spray of insecticides. From their numerical results they found that an effective and optimal use of preventive measure in the population without the use of larvicide against the vector will not be beneficial if total elimination of the disease is desirable in the community. Control programs that follow these strategies can effectively reduce the spread of a vector-borne disease in the community.

Wijayaa *et al*., (2013) presented optimal control model of Aedes aegypti population dynamics concerning classification of indoor-outdoor life cycles. An optimal control based on the mosquito population dynamics regulated by the two control measures: the Temephos spraying and the thermal fogging. The basic mosquito offspring was obtained from the maximum of the modulus of all elements in the spectrum of the next generation matrix. Preliminary simulation result shows that the mosquito-free equilibrium (referred to as the trivial equilibrium) was always unstable for any choice of the constant control measures undertaken in the simulation. Whereas the constant control seems less applicable in everyday life, an optimal control was required such that the balance between minimizing the cost for the control and suppressing the trajectory of all compartments is achieved together.

From their optimal control simulation, the results show that all the controlled trajectories lied under all the associated uncontrolled trajectories after performing the optimal control. From every specific scenario of the control implementation, one needs to enhance the mass of the thermal fogging rather than the mass of the Temephos spraying during the observation time.

Lashari *et al.,* (2013) delt with a simple mathematical model for the transmission dynamics of a vector-borne disease that incorporates both direct and indirect transmission. The model was analysed using dynamical systems techniques and it revealed the backward bifurcation for some range of parameters. In such cases, the reproduction number does not describe the necessary elimination effort of disease rather the effort is described by the value of the critical parameter at the turning point. The model was extended to assess the impact of some control measures, by re-formulating it as an optimal control problem with density-dependent demographic parameters. The optimality system was derived and solved numerically to investigate the cost effective control efforts in reducing the incidence of infectious hosts and vectors. They also determined the cost effective strategies for combating the spread of a vector-borne infection in the community. By the application of Pontryagins

Maximum Principle, they performed the optimal analysis of the non-autonomous control problem considering three controls, one for mosquito-reduction strategies and the other two for personal (human) protection and blood screening, respectively. Furthermore, they minimized the number of infected hosts and the total number of vector population by using three control variables. They investigated the dynamics by an efficient numerical method based on optimal control to identify the best strategy of a vector-borne disease in order to reduce infection and prevent vector host as well as direct contacts by using three controls. The results support the hypothesis that preventive practices are very effective in reducing the incidence of infectious hosts and vectors.

Fister *et al.*, (2013) developed an optimal control framework for an ordinary differential equations model to investigate the introduction of sterile mosquitoes to reduce the incidence of mosquito-borne diseases. Existence of a solution given an optimal strategy and the optimal control was determined in association with the negative effects of the disease on the population while minimizing the cost due to this control mechanism. Numerical simulations have shown the importance of effects of the bounds on the release of sterile mosquitoes and the bounds on the likelihood of egg maturation. The optimal strategy was to maximize the use of habitat modification or insecticide. A combination of techniques leads to a more rapid elimination of the wild mosquito population.

Rodrigues *et al*., (2010) presented a model for the transmission of dengue disease. It consists of eight mutually-exclusive compartments representing the human and vector dynamics. It also includes a control parameter (adulticide spray) in order to combat the mosquito. It was very difficult to control or eliminate the Aedes aegypti mosquito because it makes adaptations to the environment and becomes resistant to natural phenomena (e.g. droughts) or human interventions (e.g. control measures). During outbreaks emergency, vector control measures can also include broad application of insecticides. It is shown that, with a steady spray campaign it is possible to reduce the number of infected humans and mosquitoes. Active monitoring and surveillance of the natural mosquito population should accompany control efforts to determine programme effectiveness.

Although many Dengue fever models have been formulated so far, regarding the effectiveness of the control strategies for Dengue fever epidemics, none has considered the effects of the use of all five controls, that is campaign aimed in educating careless individuals, reducing mosquito-human contact, removing vector breeding places, insecticide application, reducing the maturation rate from larvae to adult in Tanzania, by applying Optimal Control approach using Pontryagin's Maximum principle in order to find the best strategy or strategies to fight the disease and minimize the cost.

CHAPTER THREE

A MODEL OF DENGUE FEVER WITH TREATMENT, TEMPORARY IMMUNITY, CAREFUL AND CARELESS HUMAN SUSCEPTIBLE

3.1 Introduction

In this chapter, the Dengue fever disease mathematical model by Massawe *et al.*, (2015) which was temporal model for dengue disease with treatment will be extended to include temporary immunity and Susceptibles with different behaviour that is the dynamical system that incorporates the effects of Careful and Careless human susceptible on the transmission of Dengue fever in the society. The model is analysed to get the insight into its epidemiological and dynamical features necessary for better understanding of the spread of Dengue fever infection in a population. The epidemic threshold governing the elimination or persistence of Dengue fever epidemic is determined and studied. Then local and global asymptotically stability of disease-free and endemic equilibrium point are studied. Numerical sensitivity is carried out in which sensitivity indices of the effective reproduction number R_e to each parameter in the model is calculated to determine which parameters have high impact on R_e and should be targeted for control strategies.

3.2 Formulation of the Model

In this section, a deterministic model is developed that describes the dynamics of Dengue fever of population size N (Rodrigue *et al.* 2013). Two types of population are considered: humans and mosquito. The humans are divided into five mutually-exclusive compartments indexed by h are given by: $S_{h_1}(t)$- careful human susceptible (Individual who are aware of the disease and use protective measure), $S_{h_2}(t)$- careless human susceptible (Individual who are not aware of the disease and are not using protective measure), where the biting rate and infected with the disease for careless susceptible is higher than that of careful susceptible individual. Careless susceptible individual may change in behaviour to be careful at a rate of θ_2, $I_h(t)$- individuals capable of transmitting dengue fever disease to others; $T_h(t)$- individual who are treated and $R_h(t)$- individuals who have acquired immunity at time *t*. The total number of human is constant, which means that $N(t) = S_{h_1}(t) + S_{h_2}(t) + I_h(t) + T_h(t) + R_h(t)$.

Hence we formulate the $\mathrm{S}_{h_{1,2}}\mathrm{ITRS}_{h_1}$ model to describe the passage of individual from careful or careless susceptible class $\mathrm{S}_{h_{1,2}}(\mathrm{t})$, to the infected class $I_h(t)$, to the treatment class $T_h(t)$, to the recovery class $R_h(t)$, and then to the careful susceptible class S_{h_1}, indicating that individual lose immunity on recovery from the infection class. Careful and careless susceptible individual are obtained in the population at a constant rate of $1-\pi$ and π respectively, it assumed that immigration and emigration are not considered and then the population is homogeneous, which means that every individual of a compartment is homogeneously mixed with the other individuals.

Individual acquire dengue fever infection after infected with dengue virus from mosquito biting rate B_1 and B_2 for careful and careless susceptible with the force of infection $\lambda_1 = B_1\beta_{mh}\dfrac{I_m}{N_h}$ and $\lambda_2 = B_2\beta_{mh}\dfrac{I_m}{N_h}$ respectively, β_{mh} is infection from mosquito to human. It is assumed that $B_2 > B_1$, there is no natural protection for human and mosquito.

However infected individual can die by the disease with the rate of a ,or can move to the other class which is treatment at the rate of η_h.

Furthermore dengue fever infected individual progress for treatment at the rate of δ_h where the treated class move to the recovery class at a time t and then lose immunity at a rate of θ_1. Human classes are assumed to die naturally at a rate of μ_h. Similarly, the model has also three compartments for the mosquito (mosquitoes) indexed by m are given by: $A_m(t)$, which represents the aquatic phase of the mosquito (including egg, pupae and larvae) and the adult phase of the mosquito, with $S_m(t)$ and $I_m(t)$, susceptible and infected, respectively. It is also assumed that $N_m(t) = S_m(t) + I_m(t)$. Then also we formulate the $A_m\, S_m \mathrm{I}_m$ model to describe the passage from the aquatic phase of the mosquito $A_m(t)$, to the the adult phase of susceptible mosquito $S_m(t)$ and to the the adult phase of infected mosquito $I_m(t)$.

The eggs are obtained from either susceptible or infected mosquito at the rate of φ For which aquatic phase will mature to adult at the rate of η_A, susceptible mosquito will be infected with dengue virus after biting infected human at the rate of B_3, with the force of infection $\lambda_3 = B_3\beta_{mh}\frac{I_m}{N_h}$, where, β_{hm} is transmission probability from human to mosquito.

It also assumed that each vector has an equal probability to bite any host and there is no resistant phase, due to its short lifetime. Furthermore aquatic phase $A_m(t)$ and adult phase of mosquito can die naturally at the rate of μ_A and μ_m respectively.

Table 3. 1: Definitions of parameters

PARAMETERS	DESCRIPTIONS
N_h	Total human population
B_1, B_2 & B_3	Average daily biting (per day)with $B_2 > B_1$
β_{mh}	Transmission probability from mosquito to human (per bite)
β_{hm}	transmission probability from human to mosquito (per bite)
μ_h	Average lifespan of humans (in days)
η_h	Mean viremic period (in days)
μ_m	Average lifespan of adult mosquitoes (in days)
φ	Number of eggs at each deposit per capita (per day)
μ_A	Natural mortality of larvae (per day)
δ_h	Rate at which dengue fever infected individuals progress for treatment
η_A	Maturation rate from larvae to adult (per day)
m	Female mosquitoes per human
K	Number of larvae per human
π	Fraction of subpopulation recruited into the population.
θ_1	Rate at which recovery individuals lose immunity
θ_2	Positive change in behaviour of Careless individuals
a	Per capita disease induced death rate for humans

Considering the above considerations and assumptions, we then have the following schematic model flow diagram for dengue fever disease with treatment, temporary immunity, careful and careless human susceptible:

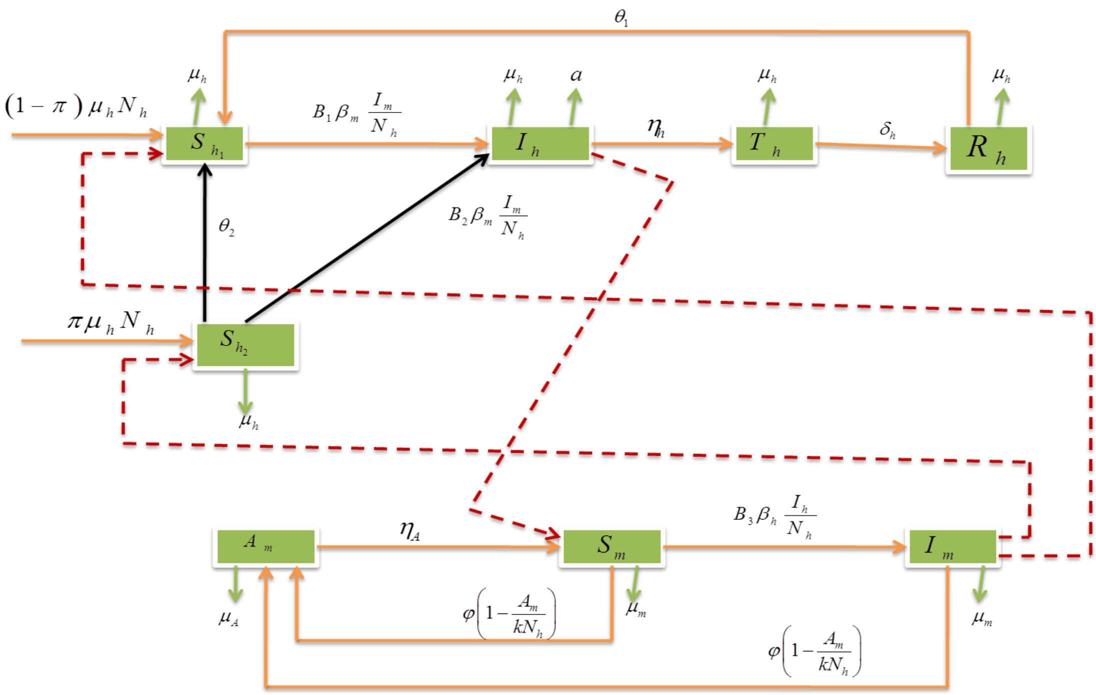


Figure 3. 1: Model flow diagram for dengue fever disease with treatment, temporary immunity, careful and careless human susceptible

From the above flow diagram, the model is described by an initial value problem with a system of eight differential equations given as follows:

$$
\begin{aligned}
\frac{dS_{h_1}}{dt} &= (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2} \\
\frac{dS_{h_2}}{dt} &= \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - (\mu_h + \theta_2)S_{h_2} \\
\frac{dI_h}{dt} &= \left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h \\
\frac{dT_h}{dt} &= \eta_h I_h - (\mu_h + \delta_h)T_h \\
\frac{dR_h}{dt} &= \delta_h T_h - (\mu_h + \theta_1)R_h
\end{aligned}
\qquad (3.1)
$$

$$\frac{dA_m}{dt} = \varphi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - (\mu_A + \eta_A)A_m$$
$$\frac{dS_m}{dt} = \eta_A A_m - \left(B_3\beta_{hm}\frac{I_h}{N_h} + \mu_m\right)S_m$$
$$\frac{dI_m}{dt} = B_3\beta_{hm}\frac{I_h}{N_h}S_m - \mu_m I_m$$

3.3 Model Analysis

The model system of equations (3.1) will be analysed qualitatively to get insight into its dynamical features which will give a better understanding of Dengue Epidemics in the society. Threshold which governs elimination or persistence of Dengue fever will be determined and studied. We begin by finding the invariant region and show that all solutions of system (3.1) are positive for all the time.

3.3.1 Positive Invariant Region of the Model

Since the model system (3.1) is Dengue fever disease model dealing with human population, we assume that all state variables and parameters of the model are positive for all $t \geq 0$. The model (3.1) will be analysed in suitable feasible region where all state variables are positive. This region is contained in $\Omega \in R_+^8$

Let $\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right) \in R_+^8$ be any solution of the system (3.1) given by $S_{h_1} + S_{h_2} + I_h + T_h + R_h \leq N_h$, $A_m \leq kN_h$ & $S_m + I_m \leq mN_h$ with non- negative initial conditions, (Rodrigues *et al.*, 2013).

Then the solutions $\left\{S_{h_1}(t), S_{h_2}(t), I_h(t), T_h(t), R_h(t), A_m(t), S_m(t), I_m(t)\right\}$ of the system (3.1) are positive for all the time.

Proof:

To prove Positive Invariant Region of the Model, we shall use all the equations of system (3.1) of human population whose total is denoted by N_h, population of aquatic phase denoted by A_m and population of mosquito whose total is denoted by N_m.

First, with regards of the population of human whose total is denoted by N_h, we have

$N_h = S_{h_1} + S_{h_2} + I_h + T_h + R_h$ and the system of differential equations is given by

$$\frac{dN_h}{dt} = \frac{dS_{h_1}}{dt} + \frac{dS_{h_2}}{dt} + \frac{dI_h}{dt} + \frac{dT_h}{dt} + \frac{dR_h}{dt}.$$

Using system of equations (3.1), we get

$$\frac{dN_h}{dt} = (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2} +$$

$$\pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2} +$$

$$\left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h +$$

$$\eta_h I_h - (\mu_h + \delta_h)T_h +$$

$$\delta_h T_h - (\mu_h + \theta_1)R_h$$

This gives

$$\frac{dN_h}{dt} \geq \mu_h N_h - \mu_h\left(S_{h_1} + S_{h_2} + I_h + T_h + R_h\right)$$

Then

$$\frac{dN_h}{dt} \geq \mu_h N_h - \mu_h N_h$$

or $$\frac{dN_h}{dt} \geq 0.$$

Consequently

$$dN_h \geq dt.$$

Integrating both sides, gives

$$N_h \geq c.$$

Since

$$N_h \geq S_{h_1} + S_{h_2} + I_h + T_h + R_h$$

it follows that

$$N_h \geq S_{h_1} + S_{h_2} + I_h + T_h + R_h \geq c.$$

Hence

$$S_{h_1} + S_{h_2} + I_h + T_h + R_h \leq N_h.$$

Secondly, with regards of the population of aquatic phase denoted by A_m, we have

$$\frac{dA_m}{dt} = \varphi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - (\mu_A + \eta_A) A_m \qquad \text{or}$$

$$\frac{dA_m}{dt} \geq -(\mu_A + \eta_A) A_m.$$

This is the first order inequality which can be solved using separation of variables

Then

$$\frac{dA_m}{A_m} \geq -\left(\mu_A + \eta_A\right) dt$$

Integrating both sides one gets

$$\ln A_m \geq -\left(\mu_A + \eta_A\right) t + \ln C$$

This is equivalent to

$$A_m\left(t\right) \geq Ce^{-\left(\mu_A + \eta_A\right)t} .$$

As $t \to 0 \ A_m\left(0\right) \geq C.$

Then

$$A_m\left(t\right) \geq A_m(0)e^{-\left(\mu_A + \eta_A\right)t}$$

As $t \to \infty, \ A_m\left(t\right) \geq 0$

Then

$$0 \leq A_m\left(t\right) \leq kN_h$$

Finally, with regards of the total population of mosquito, we have

$$N_m = S_m + \mathrm{I}_m$$

and the system of differential equations, is given by

$$\frac{dN_m}{dt} = \frac{dS_m}{dt} + \frac{dI_m}{dt} .$$

But from the system (3.1) we have

$$\frac{dS_m}{dt} = \eta_A A_m - \left(B_3 \beta_{hm} \frac{I_h}{N_h} + \mu_m \right) S_m$$

and

$$\frac{dI_m}{dt} = B_3 \beta_{hm} \frac{I_h}{N_h} S_m - \mu_m I_m .$$ Then

$$\frac{dN_m}{dt} = \eta_A A_m - \left(B_3 \beta_{hm} \frac{I_h}{N_h} + \mu_m \right) S_m + B_3 \beta_{hm} \frac{I_h}{N_h} S_m - \mu_m I_m$$

or

$$\frac{dN_m}{dt} \geq -\mu_m \left(S_m + I_m \right).$$

It follows that

$$\frac{dN_m}{dt} \geq -\mu_m N_m.$$

This is the first order inequality which can be solved using separation of variables.

Then

$$\frac{dN_m}{N_m} \geq -\mu_m dt.$$

Integrate both sides gives

$$\ln N_m \geq -\mu_m t + \ln C$$

This is equivalent to

$$N_m(t) \geq Ce^{-\mu_m t}$$

As $t \to 0$, $N_m(0) \geq C$

Then $N_m(t) \geq N_m(0) e^{-\mu_m t}$

As $t \to \infty$, $N_m(t) \geq 0$

Then

$$0 \leq N_m \leq S_m + I_m \leq mN_h$$

Therefore

$$0 \leq S_m + I_m \leq mN_h$$

Hence the host population size $S_{h_1} + S_{h_2} + I_h + T_h + R_h \le N_h$. For Aquatic phase (that includes the egg, larvae and pupae stages), the total population size $A_m \le kN_h$ as $t \to \infty$.

Finally, for vector the total population size, $S_m + I_m \le mN_h$ as $t \to \infty$

Therefore the feasible set for the model system (3.1) is given by

$$\Omega = \left\{ \begin{array}{l} (S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m) \in R_+^8 : S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m \ge 0, \\ S_{h_1} + S_{h_2} + I_h + T_h + R_h \le N_h, A_m \le kN_h, S_m + I_m \le mN_h \end{array} \right\}$$

Hence it is verified that Ω is a positively invariant set with respect to (3.1).

3.3.2 Positivity of the solutions

Since model system (3.1) is dealing with human population, it is assumed that all state variables and parameters of the model are positive for all $t \ge 0$. For the system (3.1) to be epidemiologically meaningful, we shall prove that all solutions with non-negative initial data will remain non-negative that is $S_{h_1}(0), S_{h_2}(0), I_h(0), T_h(0)$, $R_h(0)$, $A_m(0)$, $S_m(0)$ & $I_m(0)$ are non-negative. We prove by the following Lemma:

Lemma 3.1

Let $\{S_{h_1}(0) \ge 0,\ S_{h_2}(0) \ge 0,\ I_h(0) \ge 0,\ T_h(0) \ge 0,\ R_h(0) \ge 0,\ A_m(0) \ge 0,$ $S_m(0) \ge 0$ and $I_m(0) \ge 0\} \in \Omega$. Then the solution set $\{S_{h_1}(t), S_{h_2}(t), I_h(t), T_h(t),$

$R_h(t),\ A_m(t),\ S_m(t),\ I_m(t)\}$ of the model system (3.1) is positive, for all $t \geq 0$.

(Ratera *et al.,* 2012)

Proof:

To prove the Lemma, we shall use all the equations of the system (3.1).

From the 1st equation of the system (3.1) we have

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2}$$

or

$$\frac{dS_{h_1}}{dt} \leq (1-\pi)\mu_h N_h - \mu_h S_{h_1}.$$

Consequently

$$\frac{dS_{h_1}}{dt} + \mu_h S_{h_1} \leq (1-\pi)\mu_h N_h.$$

This is a first order inequality which can be solved using an integrating factor I.F:

$e^{\mu_h \int dt} = e^{\mu_h t}$. Multiply the inequality by the I.F on both sides, we get

$$e^{\mu_h t}\frac{dS_{h_1}}{dt} + e^{\mu_h t}\mu_h S_{h_1} \leq e^{\mu_h t}(1-\pi)\mu_h N_h.$$

This is equivalent to

$$d\left(e^{\mu_h t}S_{h_1}(t)\right) \leq (1-\pi)\mu_h N_h e^{\mu_h t}dt.$$

Integrate both sides yields

$$e^{\mu_h t}S_{h_1}(t) \leq (1-\pi)N_h e^{\mu_h t} + C.$$

This gives

$$S_{h_1}(t) \le (1-\pi)N_h + Ce^{-\mu_h t}.$$

As $t \to 0$, it follows that $S_{h_1}(0) \le (1-\pi)N_h + C$ or

$$S_{h_1}(0) - (1-\pi)N_h \le C.$$

Consequently

$$S_{h_1}(t) \le (1-\pi)N_h + \left(S_h(0) - (1-\pi)N_h\right)e^{-\mu_h t}$$

As $t \to \infty$ then $S_{h_1}(t) \le (1-\pi)N_h$. Therefore $0 \le s_{h_1}(t) \le (1-\pi)N_h$.

From the second equation of the system (3.1), we have

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2} \qquad \text{or}$$

$$\frac{dS_{h_2}}{dt} \le \pi\mu_h N_h - (\mu_h + \theta_2)S_{h_2}.$$

Consequently

$$\frac{dS_{h_2}}{dt} + (\mu_h + \theta_2)S_{h_2} \le \pi\mu_h N_h.$$

This is a first order inequality which can be solved using an integrating factor I.F:

$$e^{(\mu_h+\theta_2)\int dt} = e^{(\mu_h+\theta_2)t}$$

Multiply the inequality by the I.F on both sides, to get

$$e^{(\mu_h+\theta_2)t}\frac{dS_{h_2}}{dt} + e^{(\mu_h+\theta_2)t}\mu_h S_{h_1} \le e^{(\mu_h+\theta_2)t}\pi\mu_h N_h.$$

This is equivalent to

$$d\left(e^{(\mu_h+\theta_2)t}S_{h_2}(t)\right) \le \pi\mu_h N_h e^{(\mu_h+\theta_2)t}dt.$$

Integrating both sides yields

$$e^{(\mu_h+\theta_2)t} S_{h_2}(t) \le \frac{\pi\mu_h N_h}{\mu_h+\theta_2} e^{(\mu_h+\theta_2)t} + C.$$

This gives

$$S_{h_2}(t) \le \frac{\pi\mu_h N_h}{\mu_h+\theta_2} + Ce^{-(\mu_h+\theta_2)t}.$$

As $t \to 0$, it follows that $S_{h_2}(0) \le \frac{\pi\mu_h N_h}{\mu_h+\theta_2} + C$ or

$$S_{h_2}(0) - \frac{\pi\mu_h N_h}{\mu_h+\theta_2} \le C.$$

Consequently

$$S_{h_2}(t) \le \frac{\pi\mu_h N_h}{\mu_h+\theta_2} + \left(S_{h_2}(0) - \frac{\pi\mu_h N_h}{\mu_h+\theta_2} \right) e^{-(\mu_h+\theta_2)t}.$$

As $t \to \infty$ then $S_{h_2}(t) \le \frac{\pi\mu_h N_h}{\mu_h+\theta_2}$. Therefore $0 \le S_{h_2}(t) \le \frac{\pi\mu_h N_h}{\mu_h+\theta_2}$.

From the third equation of the system (3.1), we have

$$\frac{dI_h}{dt} = \left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - \left(\mu_h + \eta_h + a\right) I_h \qquad \text{or}$$

$$\frac{dI_h}{dt} \ge -\left(\mu_h + \eta_h + a\right) I_h$$

This is the first order inequality which can be solved using separation of variable.

Then

$$\frac{dI_h}{I_h} \ge -\left(\mu_h + \eta_h + a\right) dt$$

Integrating both sides gives

$$\ln I_h \ge -\left(\mu_h + \eta_h + a\right) t + \ln C,$$

which is equivalent to

$$I_h(t) \ge Ce^{-(\mu_h+\eta_h+a)t}$$

As $t \to 0$ it follows that $I_h(0) \ge C$

Consequently $I_h(t) \ge I_h(0)e^{-(\mu_h+\eta_h+a)t}$

As $t \to \infty$, it follows that $I_h(t) \ge 0$

Therefore $I_h(t) \ge 0 \quad \forall t \ge 0$

Similarly, it is shown that the remaining five equations of system (3.1) are all positive.

Therefore it is true that $S_{h_1}(t) \ge 0, S_{h_2}(t) \ge 0$, $I_h(t) \ge 0$, $T_h(t) \ge 0$, $R_h(t) \ge 0$, $A_m(t) \ge 0$, $S_m(t) \ge 0$ and $I_m(t) \ge 0$, $\forall t \ge 0$

3.4 Steady State Solutions

In this section the model system (3.1) is qualitatively analysed by determining the model equilibria, carrying out their corresponding stabilities analysis and interpreting the results. Let $E = \left(S_{h_1}^*, S_{h_2}^*, I_h^*, T_h^*, R_h^*, A_m^*, S_m^*, I_m^*\right)$ be the equilibrium point of the system (3.1). Then, setting the right hand side of system (3.1) to zero, one obtains

$$(1-\pi)\mu_h N_h - \lambda_1^* S_{h_1}^* - \mu_h S_{h_1}^* + \theta_1 R_h^* + \theta_2 S_{h_2}^* = 0 \tag{3.2}$$

$$\pi\mu_h N_h - \lambda_2^* S_{h_2}^* - \mu_h S_{h_2}^* - \theta_2 S_{h_2}^* = 0 \tag{3.3}$$

$$\left(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*\right) - \left(\mu_h + \eta_h + a\right) I_h^* = 0 \tag{3.4}$$

$$\eta_h I_h^* - \left(\mu_h + \delta_h\right) T_h^* = 0 \tag{3.5}$$

$$\delta_h T_h^* - \left(\mu_h + \theta_1\right) R_h^* = 0 \tag{3.6}$$

$$\varphi\left(1-\frac{A_m^*}{kN_h}\right)\left(S_m^*+I_m^*\right)-\left(\mu_A+\eta_A\right)A_m^*=0 \tag{3.7}$$

$$\eta_A A_m^*-\left(\lambda_3^*+\mu_m\right)S_m^*=0 \tag{3.8}$$

$$\lambda_3^* S_m^*-\mu_m I_m^*=0 \tag{3.9}$$

Forces of infections are

$$\lambda_1^*=B_1\beta_{mh}\frac{I_m^*}{N_h} \tag{3.10}$$

$$\lambda_2^*=B_2\beta_{mh}\frac{I_m^*}{N_h} \tag{3.11}$$

$$\lambda_3^*=B_3\beta_{hm}\frac{I_h^*}{N_h} \tag{3.12}$$

We compute all state variables of dengue fever disease model in terms of the force of infection λ^*.

From (3.2) we have

$$S_{h_1}^*=\frac{\left(1-\pi\right)\mu_h N_h+\theta_1 R_h^*+\theta_2 S_{h_2}^*}{\left(\lambda_1^*+\mu_h\right)} \tag{3.13}$$

Then we substitute (3.6) into (3.13).

from (3.6) $R_h^*=\dfrac{\delta_h T_h^*}{\left(\mu_h+\theta_1\right)}$

where

$T_h^*=\dfrac{\eta_h I_h^*}{\left(\mu_h+\delta_h\right)}$ from (3.5) and

$$I_h^* = \frac{\left(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*\right)}{\left(\mu_h + \eta_h + a\right)} \qquad \text{from (3.4)}$$

Therefore $R_h^* = \dfrac{\delta_h \eta_h \left(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*\right)}{\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)}$

Consequently $S_{h_1}^* = \dfrac{\left(1-\pi\right)\mu_h N_h + \theta_1 \dfrac{\delta_h \eta_h \left(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*\right)}{\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)} + \theta_2 S_{h_2}^*}{\left(\lambda_1^* + \mu_h\right)}$

or

$$S_{h_1}^* = \frac{\begin{array}{c}\left(1-\pi\right)\mu_h N_h \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right) + \theta_1 \delta_h \eta_h \lambda_1^* S_{h_1}^* + \theta_1 \delta_h \eta_h \lambda_2^* S_{h_2}^* \\ + \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)\theta_2 S_{h_2}^*\end{array}}{\left(\lambda_1^* + \mu_h\right)\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)}$$

It follows that

$$\begin{aligned}&\left(\left(\lambda_1^* + \mu_h\right)\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right) - \theta_1 \delta_h \eta_h \lambda_1^*\right) S_{h_1}^* = \\ &\quad \left(1-\pi\right)\mu_h N_h \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right) + \theta_1 \delta_h \eta_h \lambda_2^* S_{h_2}^* \\ &\quad + \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)\theta_2 S_{h_2}^*.\end{aligned}$$

Then

$$S_{h_1}^* = \frac{\begin{array}{c}\left(1-\pi\right)\mu_h N_h \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right) + \theta_1 \delta_h \eta_h \lambda_2^* S_{h_2}^* \\ + \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)\theta_2 S_{h_2}^*\end{array}}{\left(\left(\lambda_1^* + \mu_h\right)\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right) - \theta_1 \delta_h \eta_h \lambda_1^*\right)}$$

Substitute $\quad S_{h_2}^* = \dfrac{\pi \mu_h N_h}{\left(\lambda_2^* + \mu_h + \theta_2\right)}$ from (3.3) to obtain

$$S_{h_1}^* = \frac{\begin{array}{c}\left(1-\pi\right)\mu_h N_h \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)\left(\lambda_2^* + \mu_h + \theta_2\right) + \theta_1 \delta_h \eta_h \pi \mu_h N_h \lambda_2^* \\ + \left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right)\theta_2 \pi \mu_h N_h\end{array}}{\left(\lambda_2^* + \mu_h + \theta_2\right)\left(\left(\lambda_1^* + \mu_h\right)\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\mu_h + \eta_h + a\right) - \theta_1 \delta_h \eta_h \lambda_1^*\right)}$$

Let $f_1 = (1-\pi)\mu_h N_h(\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)$ $\qquad f_2 = \mu_h+\theta_2$

$f_3 = (\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)\theta_2\pi\mu_h N_h$

$f_4 = (\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)$ $\; f_5 = \theta_1\delta_h\eta_h\pi\mu_h N_h$

Then

$$S_{h_1}^* = \frac{f_1\left(\lambda_2^*+f_2\right)+f_5\lambda_2^*+f_3}{\left(\lambda_2^*+f_2\right)\left(\left(\lambda_1^*+\mu_h\right)f_4-\theta_1\delta_h\eta_h\lambda_1^*\right)} \tag{3.14}$$

Then from (3.3)

$$S_{h_2}^* = \frac{\pi\mu_h N_h}{\left(\lambda_2^*+f_2\right)} \tag{3.15}$$

From equation (3.4)

$$I_h^* = \frac{\left(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*\right)}{\left(\mu_h+\eta_h+a\right)}$$

we substitute (3.14) and (3.15) to obtain

$$I_h^* = \frac{\left(\lambda_1^*\dfrac{f_1\left(\lambda_2^*+f_2\right)+f_5\lambda_2^*+f_3}{\left(\lambda_2^*+f_2\right)\left(\left(\lambda_1^*+\mu_h\right)f_4-\theta_1\delta_h\eta_h\lambda_1^*\right)}+\lambda_2^*\dfrac{\pi\mu_h N_h}{\left(\lambda_2^*+f_2\right)}\right)}{\left(\mu_h+\eta_h+a\right)}$$

It follows that

$$I_h^* = \frac{\lambda_1^*\left(f_1\left(\lambda_2^*+f_2\right)+f_5\lambda_2^*+f_3\right)+\left(\left(\lambda_1^*+\mu_h\right)f_4-\theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h}{\left(\lambda_2^*+f_2\right)\left(\left(\lambda_1^*+\mu_h\right)f_4-\theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h+\eta_h+a\right)} \tag{3.16}$$

Then substituting (3.16) into (3.5) yields

$$T_h^* = \frac{\eta_h I_h^*}{\left(\mu_h+\delta_h\right)}$$

It follows that

$$T_h^* = \frac{\lambda_1^*\eta_h\left(f_1\left(\lambda_2^* + f_2\right) + f_5\lambda_2^* + f_3\right) + \left(\left(\lambda_1^* + \mu_h\right)f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h}{\left(\mu_h + \delta_h\right)\left(\lambda_2^* + f_2\right)\left(\left(\lambda_1^* + \mu_h\right)f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h + \eta_h + a\right)} \tag{3.17}$$

substitute (3.17) into (3.6) to obtain

$$R_h^* = \frac{\delta_h T_h^*}{\left(\mu_h + \theta_1\right)}.$$

Consequently

$$R_h^* = \frac{\lambda_1^*\delta_h\eta_h\left(f_1\left(\lambda_2^* + f_2\right) + f_5\lambda_2^* + f_3\right) + \left(\left(\lambda_1^* + \mu_h\right)f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h}{\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\lambda_2^* + f_2\right)\left(\left(\lambda_1^* + \mu_h\right)f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h + \eta_h + a\right)} \tag{3.18}$$

Then from (3.8) and (3.9) we have

$$S_m^* = \frac{\eta_A A_m^*}{\left(\lambda_3^* + \mu_m\right)} \tag{3.19}$$

$I_m^* = \dfrac{\lambda_3^* S_m^*}{\mu_m}$ which is equivalent to

$$I_m^* = \frac{\lambda_3^*\eta_A A_m^*}{\mu_m\left(\lambda_3^* + \mu_m\right)}. \tag{3.20}$$

Substitute (3.19) and (3.20) into (3.7) to obtain

$$\varphi\left(1 - \frac{A_m^*}{kN_h}\right)\left(\frac{\eta_A A_m^*}{\left(\lambda_3^* + \mu_m\right)} + \frac{\lambda_3^*\eta_A A_m^*}{\mu_m\left(\lambda_3^* + \mu_m\right)}\right) - \left(\mu_A + \eta_A\right)A_m^* = 0$$

or

$$\varphi\left(1 - \frac{A_m^*}{kN_h}\right)\left(\frac{\mu_m\eta_A A_m^* + \lambda_3^*\eta_A A_m^*}{\mu_m\left(\lambda_3^* + \mu_m\right)}\right) = \left(\mu_A + \eta_A\right)A_m^*.$$

It follows that

$$\frac{A_m^*}{kN_h} = \frac{\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right) - \left(\mu_A + \eta_A\right)\mu_m\left(\lambda_3^* + \mu_m\right)}{\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right)}.$$

Consequently

$$A_m^* = \frac{kN_h\left(\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right) - \left(\mu_A + \eta_A\right)\mu_m\left(\lambda_3^* + \mu_m\right)\right)}{\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right)} \tag{3.21}$$

Now we substitute (3.21) into (3.19) and (3.20) which gives

$$S_m^* = \frac{kN_h\eta_A\left(\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right) - \left(\mu_A + \eta_A\right)\mu_m\left(\lambda_3^* + \mu_m\right)\right)}{\varphi\left(\lambda_3^* + \mu_m\right)\left(\mu_m\eta_A + \lambda_3^*\eta_A\right)} \tag{3.22}$$

$$I_m^* = \frac{kN_h\lambda_3^*\eta_A\left(\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right) - \left(\mu_A + \eta_A\right)\mu_m\left(\lambda_3^* + \mu_m\right)\right)}{\varphi\mu_m\left(\lambda_3^* + \mu_m\right)\left(\mu_m\eta_A + \lambda_3^*\eta_A\right)} \tag{3.23}$$

Furthermore by substituting (3.16) and (3.23) into (3.10) to (3.12) yields

$$\lambda_1^* = \frac{B_1\beta_{mh}kN_h\lambda_3^*\eta_A\left(\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right) - \left(\mu_A + \eta_A\right)\mu_m\left(\lambda_3^* + \mu_m\right)\right)}{N_h\varphi\mu_m\left(\lambda_3^* + \mu_m\right)\left(\mu_m\eta_A + \lambda_3^*\eta_A\right)} \tag{3.24}$$

$$\lambda_2^* = \frac{B_2\beta_{mh}kN_h\lambda_3^*\eta_A\left(\varphi\left(\mu_m\eta_A + \lambda_3^*\eta_A\right) - \left(\mu_A + \eta_A\right)\mu_m\left(\lambda_3^* + \mu_m\right)\right)}{N_h\varphi\mu_m\left(\lambda_3^* + \mu_m\right)\left(\mu_m\eta_A + \lambda_3^*\eta_A\right)} \tag{3.25}$$

$$\lambda_3^* = \frac{\lambda_1^* B_3\beta_{hm}\left(f_1\left(\lambda_2^* + f_2\right) + f_5\lambda_2^* + f_3\right) + \left(\left(\lambda_1^* + \mu_h\right)f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h}{N_h\left(\lambda_2^* + f_2\right)\left(\left(\lambda_1^* + \mu_h\right)f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h + \eta_h + a\right)} \tag{3.26}$$

We substitute (3.24) and (3.25) into (3.26) to get

$$\lambda_3^*\left(A\lambda_3^{*2} + B\lambda_3^* + C\right) = 0$$

It follows that

$\lambda_3^* = 0$ or $\qquad \left(A\lambda_3^{*2} + B\lambda_3^* + C\right) = 0$

where

$$A = -\varphi f_2 N_h\left(a + \eta_h + \mu_h\right)\mu_m\left(\varphi f_4\mu_h\mu_m + kB_1\beta_{mh}\left(f_4 - \delta_h\eta_h\theta_1\right)\left(\varphi\eta_A\right.\right.$$

$$-\left(\eta_A+\mu_A\right)\mu_m\right)\Big)-kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)\Big(\varphi\eta_A-\left(\eta_A+\mu_A\right)$$

$$\mu_m\Big)\Big(\varphi f_4\mu_h\mu_m+kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)\Big(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\Big)\Big)$$

$$B=-\varphi^2 f_2 f_4 N_h\mu_h\left(a+\eta_h+\mu_h\right)\mu_m^3+k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\mu_m\left(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\right)-$$

$$k\varphi B_2 f_4 N_h\beta_{mh}\mu_h\left(a+\eta_h+\mu_h\right)\mu_m^2\left(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\right)-\varphi f_2 N_h\left(a+\eta_h+\mu_h\right)$$

$$\mu_m^2\Big(\varphi f_4\mu_h\mu_m+kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)\Big(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\Big)\Big)+kB_1\beta_{mh}\left(\varphi\eta_A-\right.$$

$$\left.\left(\eta_A+\mu_A\right)\mu_m\right)\Big(\varphi B_3\left(f_1 f_2+f_3\right)\beta_{hm}\mu_m+kB_2\beta_{mh}\Big(B_3\left(f_1+f_5\right)\beta_{hm}+\pi N_h\left(f_4-\right.$$

$$\left.\delta_h\eta_h\theta_1\right)\mu_h\Big)\Big(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\Big)\Big)$$

$$C=-\varphi^2 f_2 f_4 N_h\mu_h\left(a+\eta_h+\mu_h\right)\mu_m^4+k\varphi B_1B_3\left(f_1f_2+f_3\right)\beta_{hm}\beta_{mh}\mu_m^2\left(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\right)$$

$$+k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\mu_m^2\left(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\right)$$

Hence the disease free equilibrium is obtained when $\lambda_3^*=0$ and endemic when

$$A\lambda_3^{*2}+B\lambda_3^*+C=0 \tag{3.27}$$

For disease free equilibrium $\lambda_3^*=0$

It follows that from (3.23) $I_m^*=0.$ Then from (3.24) and (3.25) we obtain

$\lambda_1^*=0$ and $\lambda_2^*=0.$

Thus from (3.14) to (3.18) and then (3.21) to (3.23) we have

$$S_{h_1}^*=\frac{\left(1-\pi\right)N_h\left(\mu_h+\theta_2\right)+\theta_2\pi N_h}{\mu_h+\theta_2},\ S_{h_2}^*=\frac{\pi\mu_h N_h}{\left(\mu_h+\theta_2\right)}\ ,I_h^*=0,\ T_h^*=0,\ R_h^*=0,$$

$$A_m^*=\frac{kN_hq}{\eta_A\varphi},\ \ S_m^*=\frac{kN_hq}{\varphi\mu_m}\ \text{ and } I_m^*=0$$

Where $q=\left(\eta_A\varphi-\left(\mu_A+\eta_A\right)\mu_m\right).$

Therefore the Disease Free Equilibrium (DFE) denoted by E_0 of the system (3.1) is given by $E_0 = \left(S_{h_1}(t), S_{h_2}(t), 0,0,0, A_m(t), S_m(t), 0\right) =$

$$\left(\frac{(1-\pi)N_h(\mu_h+\theta_2)+\theta_2\pi N_h}{\mu_h+\theta_2}, \frac{\pi\mu_h N_h}{\mu_h+\theta_2}, 0,0,0, \frac{kN_h q}{\eta_A\varphi}, \frac{kN_h q}{\varphi\mu_m}, 0\right) \tag{3.28}$$

3.5 The effective Reproduction Number R_e

The dynamics of dengue fever disease is determined by the basic reproduction number R_0 which is a key concept and is defined as the average number of secondary infection arising from a single infected individual introduced into the susceptible class during its entire infectious period in a totally susceptible population without intervention while effective reproduction number R_e is defined as the average number of secondary infection arising from a single infected individual introduced into the susceptible class during its entire infectious period in a totally susceptible population with (Treatment) intervention (Driessche and Watmough, 2002, Lashari *et al*., 2013), for if $R_e<1$ the result is disease free-equilibrium and if $R_e>1$ it means that there exists endemic equilibrium point. The model system of equations (3.1) will be analysed qualitatively to get a better understanding of the effects of treated individual, careful and Careless human Susceptibles of Dengue fever disease.

The effective reproduction number of the model (3.1) R_e is calculated by using the next generation matrix of an ODE (Driessche and Watmough, 2002). Using the approach of Driessche and Watmough (2002), R_e is obtaining by taking the largest (dominant) Eigen value (spectral radius) of $\left[\frac{\partial F_i(E_0)}{\partial X_j}\right]\left[\frac{\partial V_i(E_0)}{\partial X_j}\right]^{-1}$,

where, F_i is the rate of appearance of new infection in compartment i , V_i^{+} is the transfer of individuals out of the compartment i by all other means and E_0 is the disease free equilibrium.

$$F_i = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} \\ B_3\beta_{hm}\frac{I_h}{N_h}S_m \end{bmatrix}.$$

Using the linearization method, the associated matrix at DFE is given by

$$\mathbf{F} = \begin{pmatrix} \frac{\partial F_1}{\partial I_h}(E_0) & \frac{\partial F_1}{\partial I_m}(E_0) \\ \frac{\partial F_2}{\partial I_h}(E_0) & \frac{\partial F_2}{\partial I_m}(E_0) \end{pmatrix}.$$

This implies that

$$\mathbf{F} = \begin{pmatrix} 0 & \frac{\left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}}{N_h} \\ B_3\beta_{hm}\frac{S_m}{N_h} & 0 \end{pmatrix}$$

with $S_{h_1} = \frac{(1-\pi)N_h(\mu_h+\theta_2)+\theta_2\pi N_h}{\mu_h+\theta_2}$, $S_{h_2} = \frac{\pi\mu_h N_h}{\mu_h+\theta_2}$ and $S_m = \frac{kN_h q}{\varphi\mu_m}$. Then

$$\mathbf{F} = \begin{pmatrix} 0 & \left(\frac{B_1(1-\pi)(\mu_h+\theta_2)+\theta_2\pi}{\mu_h+\theta_2} + B_2\frac{\pi\mu_h}{\mu_h+\theta_2}\right)\beta_{mh} \\ \frac{B_3\beta_{hm}kq}{\varphi\mu_m} & 0 \end{pmatrix}$$

The transfer of individuals out of the compartment i is given by

$$\mathbf{V}_i = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} (\mu_h + \eta_h + a) I_h \\ \mu_m I_m \end{bmatrix}$$

Using the linearization method, the associated matrix at DFE is given by,

$$\mathbf{V} = \begin{pmatrix} \frac{\partial V_1}{\partial I_h}(E_0) & \frac{\partial V_1}{\partial I_m}(E_0) \\ \frac{\partial V_2}{\partial I_h}(E_0) & \frac{\partial V_2}{\partial I_m}(E_0) \end{pmatrix}.$$

This gives $\mathbf{V} = \begin{pmatrix} \mu_h + \eta_h + a & 0 \\ 0 & \mu_m \end{pmatrix}$

with $\mathbf{V}^{-1} = \begin{pmatrix} \frac{1}{\mu_h + \eta_h + a} & 0 \\ 0 & \frac{1}{\mu_m} \end{pmatrix}$

Therefore

$$\mathbf{FV}^{-1} = \begin{pmatrix} 0 & \left(\frac{B_1(1-\pi)(\mu_h + \theta_2) + \theta_2 \pi}{\mu_h + \theta_2} + \frac{B_2 \pi \mu_h}{\mu_h + \theta_2} \right) \beta_{mh} \\ \frac{B_3 \beta_{hm} kq}{\varphi \mu_m} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_h + \eta_h + a} & 0 \\ 0 & \frac{1}{\mu_m} \end{pmatrix} \tag{3.29}$$

The eigenvalues of the equation (3.29) are obtained by solving

$$\det\left(\mathbf{FV}^{-1} - \mathbf{I}\lambda\right) = \det \begin{pmatrix} 0 - \lambda & \left(\frac{B_1 \beta_{mh} (1-\pi)(\mu_h + \theta_2) + \theta_2 \pi \beta_{mh} + B_2 \beta_{mh} \pi \mu_h}{(\mu_h + \theta_2) \mu_m} \right) \\ \frac{kqB_3 \beta_{hm}}{\varphi \mu_m (\mu_h + \eta_h + a)} & 0 - \lambda \end{pmatrix} = 0$$

This gives

$$\lambda^2 = \left(\frac{kqB_3 \beta_{hm}}{\varphi \mu_m (\mu_h + \eta_h + a)} \right) \left(\frac{B_1 \beta_{mh} (1-\pi)(\mu_h + \theta_2) + \theta_2 \pi \beta_{mh} + B_2 \beta_{mh} \pi \mu_h}{(\mu_h + \theta_2) \mu_m} \right)$$

consequently

$$\lambda_1 = \sqrt{\left(\frac{kqB_3\beta_{hm}}{\varphi\mu_m\left(\mu_h+\eta_h+a\right)}\right)\left(\frac{B_1\beta_{mh}\left(1-\pi\right)\left(\mu_h+\theta_2\right)+\theta_2\pi\beta_{mh}+B_2\beta_{mh}\pi\mu_h}{\left(\mu_h+\theta_2\right)\mu_m}\right)} \text{ or}$$

$$\lambda_2 = -\sqrt{\left(\frac{kqB_3\beta_{hm}}{\varphi\mu_m\left(\mu_h+\eta_h+a\right)}\right)\left(\frac{B_1\beta_{mh}\left(1-\pi\right)\left(\mu_h+\theta_2\right)+\theta_2\pi\beta_{mh}+B_2\beta_{mh}\pi\mu_h}{\left(\mu_h+\theta_2\right)\mu_m}\right)}$$

It follows that the effective Reproductive number which is given by the largest Eigenvalue for model system (3.1) denoted by R_e is given

$$R_e = \sqrt{\left(\frac{kqB_3\beta_{hm}}{\varphi\mu_m\left(\mu_h+\eta_h+a\right)}\right)\left(\frac{B_1\beta_{mh}\left(1-\pi\right)\left(\mu_h+\theta_2\right)+\theta_2\pi\beta_{mh}+B_2\beta_{mh}\pi\mu_h}{\left(\mu_h+\theta_2\right)\mu_m}\right)}$$

But $q=-\left(\left(\mu_A+\eta_A\right)\mu_m-\eta_A\varphi\right)$.

Then

$$R_e = \sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}\left(\left(\mu_A+\eta_A\right)\mu_m-\eta_A\varphi\right)\left(B_1\left(1-\pi\right)\left(\mu_h+\theta_2\right)+\theta_2\pi+B_2\pi\mu_h\right)}{\varphi\mu_m^2\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)}}$$

or

$$R_e = \sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m^2\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)}} \tag{3.30}$$

Where

$$t=\left(\left(\mu_A+\eta_A\right)\mu_m-\eta_A\varphi\right)\left(B_1\left(1-\pi\right)\left(\mu_h+\theta_2\right)+\theta_2\pi+B_2\pi\mu_h\right)$$

Model System (3.1) has infection-free equilibrium E_0 if $R_e<1$, otherwise endemic equilibrium exists.

3.6 Local Stability of Disease Free Equilibrium Point

The disease free of the nonlinear model system (3.1) is given by

$$E_0 = \left(S_{h_1}(t), S_{h_2}(t), 0,0,0, A_m(t), S_m(t), 0\right) =$$

$$\left(\frac{(1-\pi)N_h(\mu_h+\theta_2)+\theta_2\pi N_h}{\mu_h+\theta_2}, \frac{\pi\mu_h N_h}{\mu_h+\theta_2}, 0,0,0, \frac{kN_h q}{\eta_A\varphi}, \frac{kN_h q}{\varphi\mu_m}, 0\right) \quad \text{from (2.28)}$$

Theorem 3.1:

The disease free equilibrium of the Dengue fever disease of model system (3.1) is locally asymptotically stable if $R_e < 1$ and is unstable if $R_e > 1$, that is Dengue fever disease can die out from the community if $\mathrm{R_e} < 1$, and can persist in the community if $R_e > 1$. Local stability of DFE point is determined by the variational matrix $\mathbf{J}_{E_0}$ of the nonlinear system (3.1) corresponding to E_0 as follows,

Let

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2} = Q_1\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2} = Q_2\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dI_h}{dt} = \left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h = Q_3\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dT_h}{dt} = \eta_h I_h - (\mu_h + \delta_h)T_h = Q_4\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dR_h}{dt} = \delta_h T_h - (\mu_h + \theta_1)R_h = Q_5\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right) \qquad (3.31)$$

$$\frac{dA_m}{dt} = \varphi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - (\mu_A + \eta_A)A_m = Q_6\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dS_m}{dt}=\eta_A A_m-\left(B_3\beta_{hm}\frac{I_h}{N_h}+\mu_m\right)S_m=Q_7\left(S_{h_1},S_{h_2},I_h,T_h,R_h,A_m,S_m,I_m\right)$$

$$\frac{dI_m}{dt}=B_3\beta_{hm}\frac{I_h}{N_h}S_m-\mu_m I_m=Q_8\left(S_{h_1},S_{h_2},I_h,T_h,R_h,A_m,S_m,I_m\right)$$

It follows that,

$$\mathbf{J}_{E_0}=\begin{bmatrix}
\frac{\partial Q_1}{\partial S_{h_1}}(E_0) & \frac{\partial Q_1}{\partial S_{h_2}}(E_0) & \frac{\partial Q_1}{\partial I_h}(E_0) & \frac{\partial Q_1}{\partial T_h}(E_0) & \frac{\partial Q_1}{\partial R_h}(E_0) & \frac{\partial Q_1}{\partial A_m}(E_0) & \frac{\partial Q_1}{\partial S_m}(E_0) & \frac{\partial Q_1}{\partial I_m}(E_0) \\
\frac{\partial Q_2}{\partial S_{h_1}}(E_0) & \frac{\partial Q_2}{\partial S_{h_2}}(E_0) & \frac{\partial Q_2}{\partial I_h}(E_0) & \frac{\partial Q_2}{\partial T_h}(E_0) & \frac{\partial Q_2}{\partial R_h}(E_0) & \frac{\partial Q_2}{\partial A_m}(E_0) & \frac{\partial Q_2}{\partial S_m}(E_0) & \frac{\partial Q_2}{\partial I_m}(E_0) \\
\frac{\partial Q_3}{\partial S_{h_1}}(E_0) & \frac{\partial Q_3}{\partial S_{h_2}}(E_0) & \frac{\partial Q_3}{\partial I_h}(E_0) & \frac{\partial Q_3}{\partial T_h}(E_0) & \frac{\partial Q_3}{\partial R_h}(E_0) & \frac{\partial Q_3}{\partial A_m}(E_0) & \frac{\partial Q_3}{\partial S_m}(E_0) & \frac{\partial Q_3}{\partial I_m}(E_0) \\
\frac{\partial Q_4}{\partial S_{h_1}}(E_0) & \frac{\partial Q_4}{\partial S_{h_2}}(E_0) & \frac{\partial Q_4}{\partial I_h}(E_0) & \frac{\partial Q_4}{\partial T_h}(E_0) & \frac{\partial Q_4}{\partial R_h}(E_0) & \frac{\partial Q_4}{\partial A_m}(E_0) & \frac{\partial Q_4}{\partial S_m}(E_0) & \frac{\partial Q_4}{\partial I_m}(E_0) \\
\frac{\partial Q_5}{\partial S_{h_1}}(E_0) & \frac{\partial Q_5}{\partial S_{h_2}}(E_0) & \frac{\partial Q_5}{\partial I_h}(E_0) & \frac{\partial Q_5}{\partial T_h}(E_0) & \frac{\partial Q_5}{\partial R_h}(E_0) & \frac{\partial Q_5}{\partial A_m}(E_0) & \frac{\partial Q_5}{\partial S_m}(E_0) & \frac{\partial Q_5}{\partial I_m}(E_0) \\
\frac{\partial Q_6}{\partial S_{h_1}}(E_0) & \frac{\partial Q_6}{\partial S_{h_2}}(E_0) & \frac{\partial Q_6}{\partial I_h}(E_0) & \frac{\partial Q_6}{\partial T_h}(E_0) & \frac{\partial Q_6}{\partial R_h}(E_0) & \frac{\partial Q_6}{\partial A_m}(E_0) & \frac{\partial Q_6}{\partial S_m}(E_0) & \frac{\partial Q_6}{\partial I_m}(E_0) \\
\frac{\partial Q_7}{\partial S_{h_1}}(E_0) & \frac{\partial Q_7}{\partial S_{h_2}}(E_0) & \frac{\partial Q_7}{\partial I_h}(E_0) & \frac{\partial Q_7}{\partial T_h}(E_0) & \frac{\partial Q_7}{\partial R_h}(E_0) & \frac{\partial Q_7}{\partial A_m}(E_0) & \frac{\partial Q_7}{\partial S_m}(E_0) & \frac{\partial Q_7}{\partial I_m}(E_0) \\
\frac{\partial Q_8}{\partial S_{h_1}}(E_0) & \frac{\partial Q_8}{\partial S_{h_2}}(E_0) & \frac{\partial Q_8}{\partial I_h}(E_0) & \frac{\partial Q_8}{\partial T_h}(E_0) & \frac{\partial Q_8}{\partial R_h}(E_0) & \frac{\partial Q_8}{\partial A_m}(E_0) & \frac{\partial Q_8}{\partial S_m}(E_0) & \frac{\partial Q_8}{\partial I_m}(E_0)
\end{bmatrix}$$

Hence the variational matrix of the nonlinear model system (3.31) is obtained as

$$\mathbf{J}_{E_0}=\begin{pmatrix}
-\mu_h & \theta_2 & 0 & 0 & \theta_1 & 0 & 0 & A \\
0 & -\theta_2-\mu_h & 0 & 0 & 0 & 0 & 0 & -\frac{\pi B_2\beta_{mh}\mu_h}{\theta_2+\mu_h} \\
0 & 0 & -a-\eta_h-\mu_h & 0 & 0 & 0 & 0 & B \\
0 & 0 & \eta_h & -\delta_h-\mu_h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_h & -\theta_1-\mu_h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\eta_A-\mu_A-\frac{q}{\mu_m} & \varphi-\frac{q}{\eta_A} & \varphi-\frac{q}{\eta_A} \\
0 & 0 & -\frac{kqB_3\beta_{hm}}{\varphi\mu_m} & 0 & 0 & \eta_A & -\mu_m & 0 \\
0 & 0 & \frac{kqB_3\beta_{hm}}{\varphi\mu_m} & 0 & 0 & 0 & 0 & -\mu_m
\end{pmatrix} \tag{3.32}$$

Where, $A=-\dfrac{B_1\beta_{hm}(\theta_2-(-1+\pi)\mu_h)}{\theta_2+\mu_h}$,

$$B=\frac{\beta_{mh}(\pi B_2\mu_h+B_1(\theta_2-(-1+\pi)\mu_h))}{\theta_2+\mu_h}$$

Thus the stability of the disease free equilibrium point is clarified by studying the behaviour of $\mathbf{J}_{E_0}$ in which for local stability of DFE we seek for its all eigenvalues to have negative real parts. It follows that, the characteristic function of the matrix (3.32) with λ being the eigenvalues of the Jacobian matrix, $\mathbf{J}_{E_0}$, by using Mathematica software the Jacobian matrix has the following values:

$$\lambda_1 = -\mu_h, \quad \lambda_2 = -\theta_2 - \mu_h$$

The other eigenvalues are given as

$$\lambda_3 = -\frac{1}{2}\left(a + \eta_h + \mu_h + \mu_m + \frac{1}{\sqrt{\varphi}\sqrt{\theta_2 + \mu_h}\sqrt{\mu_m}}\left(\sqrt{\sigma}\right)\right)$$ when $\sqrt{\sigma}$ is not a real number

$$\lambda_4 = -\delta_h - \mu_h, \qquad \lambda_5 = -\theta_1 - \mu_h$$

$$\lambda_6 = -\frac{1}{2}\left(a + \eta_h + \mu_h + \mu_m - \frac{1}{\sqrt{\varphi}\sqrt{\theta_2 + \mu_h}\sqrt{\mu_m}}\left(\sqrt{\sigma}\right)\right)$$ when $\sqrt{\sigma}$ is not a real number

$$\lambda_7 = -\frac{1}{2\mu_m}(q + \mu_m(\eta_A + \mu_A + \mu_m) + \sqrt{\alpha})$$ when $\sqrt{\alpha}$ is not a real number and

finally

$$\lambda_8 = -\frac{1}{2\mu_m}\left(q + \mu_m(\eta_A + \mu_A + \mu_m) - \sqrt{\alpha}\right)$$ when $\sqrt{\alpha}$ is not a real number

Where, $\sigma = 4k\pi q B_2 B_3 \beta_{hm} \beta_{mh} \mu_h + 4kqB_1 B_3 \beta_{hm} \beta_{mh} \left(\theta_2 - (-1 + \pi)\mu\right)_h$

$$+\varphi\left(\theta_2 + \mu_h\right)\left(a + \eta_h + \mu_h - \mu_m\right)^2 \mu_m$$

$$\alpha = q^2 + 2q\left(\eta_A + \mu_A\right)\mu_m + \left(-6q + \eta_A^2 + \mu_A^2 + 2\eta_A\left(2\varphi + \mu_A\right)\right)\mu_m^2 - 2\left(\eta_A + \mu_A\right)\mu_m^3 + \mu_m^4$$

Therefore the system is stable since all the eight eigenvalues are negative. This implies that at $R_e < 1$ the Disease-free Equilibrium point is locally asymptotically stable.

3.7 Global Stability of Disease Free Equilibrium Point

In this section, we adopt the idea of Ozair *et al*., (2013), to analyse the global behaviour of the equilibria for system (3.1). The following theorem provides the global property of the disease free equilibrium E_0 of the system. The results are obtained by means of Lyapunov function.

Theorem 3.2: If $R_e \leq 1$, then the infection-free equilibrium is globally asymptotically stable in the interior of Ω

Proof:

To determine the global stability of the disease-free equilibrium point, we construct the following Lyapunov function:

$$L(t) = -kB_3\beta_{hm}tI_h(t) + \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m I_m(t) \qquad (3.33)$$

Calculating the time derivative of L along (3.33), we obtain

$$L^{'}(t) = -kB_3\beta_{hm}tI_h^{'}(t) + \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m I_m^{'}(t)$$

Then we substitute $I_h^{'}(t)$ & $I_m^{'}(t)$ from system (3.1) to obtain

$$L^{'}(t) = -kB_3\beta_{hm}t\left(\left(B_1S_{h_1} + B_2S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h\right) +$$

$$\varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m\left(B_3\beta_{hm}\frac{I_h}{N_h}S_m - \mu_m I_m\right)$$

Consequently

$$L^{'}(t) = \varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m\mu_m I_m\left(\frac{-kB_3\beta_{hm}t\left(B_1S_{h_1} + B_2S_{h_2}\right)\beta_{mh}}{N_h\varphi(\mu_h + \eta_h + a)(\mu_h + \theta_2)\mu_m\mu_m} - 1\right)$$

$$(\mu_h + \eta_h + a)B_3\beta_{hm}\left(ktI_h + \varphi(\mu_h + \theta_2)\mu_m\frac{I_h}{N_h}S_m\right)$$

But $R_e^2 = \dfrac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m\left(\mu_h+\theta_2\right)\mu_m\left(\mu_h+\eta_h+a\right)}$. Then

$$L^{'}(t)=\varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\,\mu_m\mu_m I_m\left(\frac{\left(B_1S_{h_1}+B_2S_{h_2}\right)}{N_h}R_e^2-1\right)$$

$$-\frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m\left(\mu_h+\theta_2\right)\mu_m R_e^2}\left(ktI_h+\varphi(\mu_h+\theta_2)\,\mu_m\frac{I_h}{N_h}S_m\right)$$

implying that

$$L^{'}(t)=\varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\,\mu_m\mu_m I_m\left(\sqrt{f}R_e+1\right)\left(\sqrt{f}R_e-1\right)$$

$$-\frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m\left(\mu_h+\theta_2\right)\mu_m R_e^2}\left(ktI_h+\varphi(\mu_h+\theta_2)\,\mu_m\frac{I_h}{N_h}S_m\right)$$

Therefore

$$L^{'}(t)=-\varphi(\mu_h+\eta_h+a)(\mu_h+\theta_2)\,\mu_m\mu_m I_m\left(\sqrt{f}R_e+1\right)\left(1-\sqrt{f}R_e\right)$$

$$-\frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\mu_m\left(\mu_h+\theta_2\right)\mu_m R_e^2}\left(ktI_h+\varphi(\mu_h+\theta_2)\,\mu_m\frac{I_h}{N_h}S_m\right)$$

Where $f=\dfrac{\left(B_1S_{h_1}+B_2S_{h_2}\right)}{N_h}$

Thus, $L'(t)$ is negative if $R_e \le 1$ and $L' = 0$ if and only if $I_h = I_m = 0$ is reduced to the DFE. Consequently, the largest compact invariant set in $\{(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m) \in \Omega,\ L' = 0\}$ when $R_e \le 1$ is the singleton $\{E_0\}$. Hence, by LaSalle's invariance principle it implies that "E_0" is globally asymptotically stable in Ω (LaSalle, 1976). This completes the proof.

3.8 Existence, Local and Global Asymptotic Stability of Endemic Equilibrium

Since we are dealing with presence of dengue fever disease in human population, we can reduce system (3.1) to a 4-dimensional system by eliminating T_h, R_h, A_m and S_m respectively, in the feasible region Ω. The values of S_m can be determined by setting $S_m = mN_h - I_m$ to obtain

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + \theta_2 S_{h_2}$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - \theta_2 S_{h_2}$$

$$\frac{dI_h}{dt} = \left(B_1 S_{h_1} + B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - \left(\mu_h + \eta_h + a\right)I_h \tag{3.34}$$

$$\frac{dI_m}{dt} = B_3\beta_{hm}\frac{I_h}{N_h}\left(mN_h - I_m\right) - \mu_m I_m$$

Then we set

$$\frac{dS_{h_1}}{dt} = \frac{dS_{h_1}}{dt} = \frac{dI_h}{dt} = \frac{dI_m}{dt} = 0.$$

Then the model of system (3.34) has a unique endemic equilibrium given by

$E^* = \left(S_{h_1}^*, S_{h_2}^*, I_h^*, I_m^*\right)$ in Ω, with

$$(1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m^*}{N_h}S_{h_1}^* - \mu_h S_{h_1}^* + \theta_1 R_h^* + \theta_2 S_{h_2}^* = 0$$

$$\pi\mu_h N_h - B_2\beta_{mh}\frac{I_m^*}{N_h}S_{h_2}^* - \mu_h S_{h_2}^* - \theta_2 S_{h_2}^* = 0 \tag{3.35}$$

$$\left(B_1 S_{h_1}^* + B_2 S_{h_2}^*\right)\beta_{mh}\frac{I_m^*}{N_h} - \left(\mu_h + \eta_h + a\right)I_h^* = 0$$

$$B_3\beta_{hm}\frac{I_h^*}{N_h}\left(mN_h - I_m^*\right) - \mu_m I_m^* = 0$$

3.8.1 Existence of Endemic Equilibrium Point

Existence of endemic equilibrium depends on the quadratic equation (3.27), that is, if it has positive roots. The sign of the roots depends on the sign of A, B and C

From (3.27) we have $A\lambda_3^{*2} + B\lambda_3^* + C = 0$ where

$$A = -\varphi f_2 N_h\left(a + \eta_h + \mu_h\right)\mu_m\left(\varphi f_4\mu_h\mu_m + kB_1\beta_{mh}\left(f_4 - \delta_h\eta_h\theta_1\right)\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)\right) - kB_2 N_h\beta_{mh}\left(a + \eta_h + \mu_h\right)\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)\left(\varphi f_4\mu_h\mu_m + kB_1\beta_{mh}\left(f_4 - \delta_h\eta_h\theta_1\right)\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)\right)$$

$$B = -\varphi^2 f_2 f_4 N_h\mu_h\left(a + \eta_h + \mu_h\right)\mu_m^3 + k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\mu_m\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right) - k\varphi B_2 f_4 N_h\beta_{mh}\mu_h\left(a + \eta_h + \mu_h\right)\mu_m^2\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right) - \varphi f_2 N_h\left(a + \eta_h + \mu_h\right)\mu_m^2\left(\varphi f_4\mu_h\mu_m + kB_1\beta_{mh}\left(f_4 - \delta_h\eta_h\theta_1\right)\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)\right) + kB_1\beta_{mh}\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)\left(\varphi B_3\left(f_1 f_2 + f_3\right)\beta_{hm}\mu_m + kB_2\beta_{mh}\left(B_3\left(f_1 + f_5\right)\beta_{hm} + \pi N_h\left(f_4 - \delta_h\eta_h\theta_1\right)\mu_h\right)\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)\right)$$

$$C = -\varphi^2 f_2 f_4 N_h\mu_h\left(a + \eta_h + \mu_h\right)\mu_m^4 + k\varphi B_1 B_3\left(f_1 f_2 + f_3\right)\beta_{hm}\beta_{mh}\mu_m^2\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right) + k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\mu_m^2\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)$$

Now expressing A, B and C in terms of R_e gives

$$A = -\varphi f_2 N_h\left(a + \eta_h + \mu_h\right)\mu_m\left(\varphi f_4\mu_h\mu_m + kB_1\beta_{mh}\left(f_4 - \delta_h\eta_h\theta_1\right)\left(\varphi\eta_A - \left(\eta_A + \mu_A\right)\mu_m\right)\right) - kB_2 N_h\beta_{mh}$$

$$\left(a+\eta_h+\mu_h\right)\left(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\right)\left(\varphi f_4\mu_h\mu_m+kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)\left(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\right)\right)$$

or

$$A=-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\mu_m\left(\varphi f_4\mu_h\mu_m+kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)q\right)-$$

$$kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)q\left(\varphi f_4\mu_h\mu_m+kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)q\right)$$ then

$$A=\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\mu_m kB_1\beta_{mh}q\delta_h\eta_h\theta_1+kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)qkB_1\beta_{mh}q\delta_h\eta_h\theta_1$$

$$-kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)qkB_1\beta_{mh}qf_4-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\mu_m kB_1\beta_{mh}qf_4$$

$$-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\mu_m\varphi f_4\mu_h\mu_m-kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)q\varphi f_4\mu_h\mu_m$$

It is follows that

$$A=t_1-\left(kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)qkB_1\beta_{mh}qf_4+kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)q\varphi f_4\mu_h\mu_m\right)$$
$$-\left(a+\eta_h+\mu_h\right)f_2N_h\mu_m\varphi f_4\left(\varphi\mu_h\mu_m+kB_1\beta_{mh}q\right)$$

Then

$$A=t_1+\left(a+\eta_h+\mu_h\right)f_2\varphi N_h\mu_m f_4\left(\varphi\mu_h\mu_m+kB_1\beta_{mh}q\right)\left(\frac{-\beta_{mh}kB_2N_hq\left(kB_1\beta_{mh}q+\varphi\mu_h\mu_m\right)}{f_2N_h\mu_m\varphi\left(\varphi\mu_h\mu_m+kB_1\beta_{mh}q\right)}-1\right)$$

or

$$A=t1+t2\left(\frac{-\beta_{mh}kN_hq\left(B_2\varphi_4\mu_h\mu_m+B_2kB_1\beta_{mh}q\right)}{\varphi f_2N_h\mu_m\left(\varphi\mu_h\mu_m+kB_1\beta_{mh}q\right)}-1\right)$$ implying that

$$A=t1+t2\left(\frac{N_hqf\left(B_2\varphi_4\mu_h\mu_m+B_2kB_1\beta_{mh}q\right)\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)\mu_m}{f_2N_hf_4\left(\varphi\mu_h\mu_m+kB_1\beta_{mh}q\right)B_3\beta_{hm}t}R_e^2-1\right)$$

consequently

$$A=t_1+t_2\left(t_3R_e^2-1\right)$$

therefore

$$A=t_1+t_2\left(\sqrt{t_3}R_e+1\right)\left(\sqrt{t_3}R_e-1\right)$$

in case of B we have

$$B = -\varphi^2 f_2 f_4 N_h \mu_h (a+\eta_h+\mu_h)\mu_m^3 + k\pi\varphi B_2 f_4 N_h \beta_{mh} \mu_h^2 \mu_m (\varphi\eta_A - (\eta_A+\mu_A)\mu_m) - k\varphi B_2 f_4$$

$$N_h \beta_{mh} \mu_h (a+\eta_h+\mu_h)\mu_m^2 (\varphi\eta_A - (\eta_A+\mu_A)\mu_m) - \varphi f_2 N_h (a+\eta_h+\mu_h)\mu_m^2 (\varphi f_4 \mu_h \mu_m +$$

$$kB_1 \beta_{mh} (f_4 - \delta_h \eta_h \theta_1)(\varphi\eta_A - (\eta_A+\mu_A)\mu_m)) + kB_1 \beta_{mh} (\varphi\eta_A - (\eta_A+\mu_A)\mu_m)(\varphi B_3 (f_1 f_2$$

$$+ f_3)\beta_{hm}\mu_m + kB_2 \beta_{mh} (B_3 (f_1+f_5)\beta_{hm} + \pi N_h (f_4 - \delta_h \eta_h \theta_1)\mu_h)(\varphi\eta_A - (\eta_A+\mu_A)\mu_m))$$

or

$$B = k\pi\varphi B_2 f_4 N_h \beta_{mh} \mu_h^2 \mu_m q + \varphi f_2 N_h (a+\eta_h+\mu_h)\mu_m^2 kB_1 \beta_{mh} \delta_h \eta_h \theta_1 q + kB_1 \beta_{mh} q\varphi B_3$$

$$(f_1 f_2 + f_3)\beta_{hm}\mu_m + kB_1 \beta_{mh} qkB_2 \beta_{mh} qB_3 \beta_{hm} (f_1+f_5) + kB_1 \beta_{mh} qkB_2 \beta_{mh} q\pi N_h \mu_h f_4$$

$$-\varphi^2 f_2 f_4 N_h \mu_h (a+\eta_h+\mu_h)\mu_m^3 - k\varphi B_2 f_4 N_h \beta_{mh} \mu_h (a+\eta_h+\mu_h)\mu_m^2 q$$

$$-\varphi f_2 N_h (a+\eta_h+\mu_h)\mu_m^2 \varphi f_4 \mu_h \mu_m - \varphi f_2 N_h (a+\eta_h+\mu_h)\mu_m^2 kB_1 \beta_{mh} q f_4$$

$$-kB_1 \beta_{mh} qkB_2 \beta_{mh} q\pi N_h \delta_h \eta_h \theta_1 \mu_h$$

It follows that

$$B = t_4 + t_5 - (k\varphi B_2 f_4 N_h \beta_{mh} \mu_h (a+\eta_h+\mu_h)\mu_m^2 q + \varphi f_2 N_h (a+\eta_h+\mu_h)\mu_m^2 kB_1 \beta_{mh} q f_4$$

$$+kB_1 \beta_{mh} qkB_2 \beta_{mh} q\pi N_h \delta_h \eta_h \theta_1 \mu_h) - 2\varphi^2 f_2 f_4 N_h (a+\eta_h+\mu_h)\mu_m^3 \mu_h \qquad \text{or}$$

$$B = t_4 + t_5 - k\beta_{mh} N_h q(\varphi B_2 f_4 \mu_h (a+\eta_h+\mu_h)\mu_m^2 + \varphi f_2 (a+\eta_h+\mu_h)\mu_m^2 B_1 f_4$$

$$+B_1 qkB_2 \beta_{mh} \pi\delta_h \eta_h \theta_1 \mu_h) - 2\varphi^2 f_2 f_4 N_h (a+\eta_h+\mu_h)\mu_m^3 \mu_h$$

then

$$B = t_4 + t_5 + 2\varphi^2 f_2 f_4 N_h (a+\eta_h+\mu_h)\mu_m^3 \mu_h \times$$

$$\left(\frac{-k\beta_{mh} N_h q(\varphi B_2 f_4 \mu_h (a+\eta_h+\mu_h)\mu_m^2 + \varphi f_2 (a+\eta_h+\mu_h)\mu_m^2 B_1 f_4 + B_1 qkB_2 \beta_{mh} \pi\delta_h \eta_h \theta_1 \mu_h)}{2\varphi^2 f_2 f_4 N_h (a+\eta_h+\mu_h)\mu_m^3 \mu_h} - 1 \right)$$

or

$$B = t_4 + t_5 + 2\varphi^2 f_2 f_4 N_h (a+\eta_h+\mu_h)\mu_m^3 \mu_h \times$$

$$\left(\frac{N_h q\left(\varphi B_2 f_4 \mu_h\left(a+\eta_h+\mu_h\right)\mu_m^2+\varphi f_2\left(a+\eta_h+\mu_h\right)\mu_m^2 B_1 f_4+B_1 qkB_2\beta_{mh}\pi\delta_h\eta_h\theta_1\mu_h\right)\left(\mu_h+\theta_2\right)}{2\varphi f_2 f_4 N_h\mu_m\mu_h B_3\beta_{hm}t}R_e^2-1\right)$$

consequently

$$B=t_4+t_5+t_6\left(t_7R_e^2-1\right)$$

Therefore $\quad B=t_4+t_5+t_6\left(t_7R_e+1\right)\left(t_7R_e-1\right)$

For the case of C is

$$C=-\varphi^2 f_2 f_4 N_h\mu_h\left(a+\eta_h+\mu_h\right)\mu_m^4+k\varphi B_1B_3\left(f_1f_2+f_3\right)\beta_{hm}\beta_{mh}\mu_m^2\left(\varphi\eta_A\right.$$

$$\left.-\left(\eta_A+\mu_A\right)\mu_m\right)+k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\mu_m^2\left(\varphi\eta_A-\left(\eta_A+\mu_A\right)\mu_m\right)$$

then

$$C=-\varphi^2 f_2 f_4 N_h\mu_h\left(a+\eta_h+\mu_h\right)\mu_m^4+k\varphi B_1B_3\left(f_1f_2+f_3\right)\beta_{hm}\beta_{mh}\mu_m^2 q+k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\mu_m^2 q$$

or

$$C=k\varphi B_1B_3\left(f_1f_2+f_3\right)\beta_{hm}\beta_{mh}\mu_m^2 q+k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\mu_m^2 q+\frac{\varphi^2 f_2 f_4 N_h\mu_h\mu_m^2 k\beta_{mh}B_3\beta_{hm}t}{\left(\mu_h+\theta_2\right)\varphi R_e^2}$$

It follows that

$$C=k\beta_{mh}\varphi\mu_m^2 f_4\mu_h N_h t8\left(\frac{kB_3\beta_{hm}\beta_{mh}\varphi B_1\left(f_1f_2+f_3\right)\mu_m^2 q}{k\beta_{mh}\varphi\mu_m^2 f_4\mu_h N_h t8}+1\right)\quad \text{or}$$

$$C=k\beta_{mh}\varphi\mu_m^2 f_4\mu_h N_h t_8\left(-\frac{\varphi B_1\left(f_1f_2+f_3\right)\mu_m^2 q\left(\mu_h+\theta_2\right)\left(\mu_h+\eta_h+a\right)}{k\beta_{mh} f_4\mu_h N_h t_8 t}R_e^2+1\right)$$

consequently

$C=t_9\left(-t_{10}R_e^2+1\right) \qquad$ or $\qquad C=t_9\left(1-t_{10}R_e^2\right)$

Hence $\quad C=t_9\left(1+\sqrt{t_{10}}R_e\right)\left(1-\sqrt{t_{10}}R_e\right)$

where

$$t_1=\varphi f_2 N_h\left(a+\eta_h+\mu_h\right)\mu_m kB_1\beta_{mh}q\delta_h\eta_h\theta_1+kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)qkB_1\beta_{mh}q\delta_h\eta_h\theta_1$$

$$t_2 = (a+\eta_h+\mu_h)\varphi f_2 N_h \mu_m f_4 (\varphi\mu_h\mu_m + kB_1\beta_{mh}q)$$

$$t_3 = \frac{q(B_2\varphi_4\mu_h\mu_m + B_2kB_1\beta_{mh}q)(\mu_h+\eta_h+a)(\mu_h+\theta_2)\mu_m}{f_2(\varphi\mu_h\mu_m + kB_1\beta_{mh}q)B_3\beta_{hm}t}$$

$$t_4 = +k\pi\varphi B_2 f_4 N_h \beta_{mh}\mu_h^2\mu_m q + \varphi f_2 N_h (a+\eta_h+\mu_h)\mu_m^2 kB_1\beta_{mh}\delta_h\eta_h\theta_1 q$$

$$t_5 = +kB_1\beta_{mh}q\varphi B_3(f_1f_2+f_3)\beta_{hm}\mu_m + kB_1\beta_{mh}qkB_2\beta_{mh}qB_3\beta_{hm}(f_1+f_5) + kB_1\beta_{mh}qkB_2\beta_{mh}q\pi N_h\mu_h f_4$$

$$t_6 = 2\varphi^2 f_2 f_4 N_h (a+\eta_h+\mu_h)\mu_m^3\mu_h$$

$$t_7 = \frac{q\left(\varphi B_2 f_4\mu_h(a+\eta_h+\mu_h)\mu_m^2 + \varphi f_2(a+\eta_h+\mu_h)\mu_m^2 B_1 f_4 + B_1qkB_2\beta_{mh}\pi\delta_h\eta_h\theta_1\mu_h\right)(\mu_h+\theta_2)}{2\varphi f_2 f_4\mu_m\mu_h B_3\beta_{hm}t}$$

$$t_8 = \left(\pi B_2\mu_h q + \frac{f_2B_3\beta_{hm}t}{(\mu_h+\theta_2)R_e^2}\right) \quad , \quad t_9 = k\beta_{mh}\varphi\mu_m^2 f_4\mu_h N_h t_8 \quad , \; q = (\varphi\eta_A - (\eta_A+\mu_A)\mu_m)$$

$$t_{10} = \frac{\varphi B_1(f_1f_2+f_3)\mu_m^2 q(\mu_h+\theta_2)(\mu_h+\eta_h+a)}{k\beta_{mh} f_4\mu_h N_h t_8 t} \; , \; R_e = \sqrt{-\frac{k\beta_{mh}B_3\beta_{hm}t}{(\mu_h+\eta_h+a)\mu_m^2(\mu_h+\theta_2)\varphi}}$$

$$f_1 = (1-\pi)\mu_h N_h(\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a) \quad f_2 = (\mu_h+\theta_2)$$

$$f_5 = \theta_1\delta_h\eta_h\pi\mu_h N_h \quad f_3 = (\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)\theta_2\pi\mu_h N_h$$

$$f_4 = (\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)$$

It is observed that the coefficients A, B are non- negative if $R_e>1$ and C is positive if $R_e<1$, C is negative if $R_e>1$ so that $\lambda_3^* = \dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$ from (3.27).

Therefore the model has :

i) Unique endemic equilibrium if $R_e = 1$ implying that $C = 0$

ii) Unique endemic equilibrium if $B > 0$ and $C = 0$ or $B^2 - 4AC = 0$

iii) Two endemic equilibria if $R_e > 1$ and $C < 0$ implying that $4AC < 0$

iv) No endemic otherwise

It is clear from (i) and (ii) that the model has a unique endemic equilibrium. Further, in (iii) indicates that the model has two endemic equilibria.

This is illustrated by simulating the model equation (3.1) with parameter values in Table. 3.4.

Figure 3.2. Shows a forward bifurcation

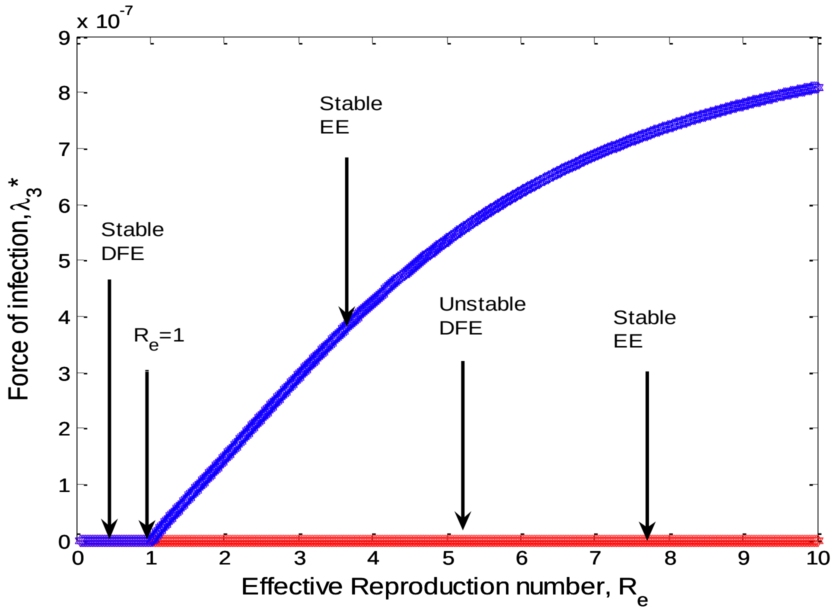


Figure 3. 2: Forward bifurcation

From figure 3.2, shows the two equilibrium points exchange stabilities depending on the value of R_e . A transcritical/forward bifurcation in the equilibrium points occur at $R_e = 1$. If $R_e < 1$, disease free equilibrium point exist i.e. no endemic equilibrium exists. But if $R_e > 1$, the endemic equilibrium exists. Thus there is a forward bifurcation because in the neighborhood of the bifurcation point, the force of infection is an increasing function of R_e as shown in figure 3.2.

3.8.2 The Endemic Equilibrium and its Stability

Here, we study the stability of the endemic equilibrium points. If $R_e > 1,$ then the host-vector model system (3.5) has a unique endemic equilibrium given by

$E^* = \left(S_{h_1}^*, S_{h_2}^*, I_h^*, I_m^*\right)$ in Ω with

$$S_{h_1}^* = \Big(m + B_3 N_h^2 \beta_{hm}\beta_{mh}\mu_h \left(B_2\left(2\theta_2 + (2-\pi)\mu_h\right) - B_1\left(\theta_2 + (1-\pi)\mu_h\right)\right) +$$

$$N_h\beta_{mh}\Big(\left(a+\eta_h+\mu_h\right)\left(B_2\left(2\theta_2+\mu_h\right) - B_1\left(\theta_2+\mu_h\right)\right)\mu_m + m + \left(2B_2 - B_1\right)$$

$$B_3\beta_{hm}\theta_1\left(\theta_2+\mu_h\right)R_h^*\Big) + \surd\Big(N_h^2\beta_{mh}^2\Big(4B_1B_2\mu_h\left(a+\eta_h+\mu_h\right)\left(\theta_2+\mu_h\right)\mu_m$$

$$\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h+\theta_2)\mu_m R_e^2} + m + B_3\beta_{hm}\left(N_h\mu_h + \theta_1 R_h^*\right)\right) + \Big(B_2\mu_h\big(-m\pi - B_3N_h\beta_{hm}\mu_h$$

$$+\left(a+\eta_h+\mu_h\right)\mu_m\big) + B_1\Big(\left(a+\eta_h+\mu_h\right)\left(\theta_2+\mu_h\right)\mu_m - m - B_3\beta_{hm}\Big(N_h\mu_h\big(\theta_2 -$$

$$\left(-1+\pi\right)\mu_h\big) + \theta_1\left(\theta_2+\mu_h\right)R_h^*\Big)\Big)\Big)^2\Big)\Big)\Big) / \left(2m + \left(B_2 - B_1\right)B_3N_h\beta_{hm}\beta_{mh}\mu_h\left(\theta_2+\mu_h\right)\right),$$

$$S_{h_2}^* = \Bigg(m + B_3N_h^2\beta_{hm}\beta_{mh}\mu_h\pi B_2\mu_h + \frac{\varphi\mu_m^2\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)R_e^2}{k\beta_{mh}t}N_h^2\beta_{mh}\mu_h B_1$$

$$\left(\theta_2 + \left(1+\pi\right)\mu_h\right) + N_h\beta_{mh}\left(\left(a+\eta_h+\mu_h\right)\left(B_2\mu_h + B_1\frac{kB_3\beta_{hm}\beta_{mh}t}{\mu_m(\mu_h+\eta_h+a)\varphi\mu_m R_e^2}\right)\mu_m + m + \right.$$

$$\left.\frac{\varphi\mu_m^2\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)R_e^2}{k\beta_{mh}t}B_1\theta_1\left(\theta_2+\mu_h\right)R_h^*\right) + \surd\Big(N_h^2\beta_{mh}^2\Big(4B_1B_2\mu_h\left(a+\eta_h+\mu_h\right)$$

$$\left(\theta_2+\mu_h\right)\mu_m\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi(\mu_h+\theta_2)\mu_m R_e^2} + m + B_3\beta_{hm}\left(N_h\mu_h + \theta_1 R_h^*\right)\right) + \Big(B_2\mu_h\big(-m\pi -$$

$$B_3N_h\beta_{hm}\mu_h + \left(a+\eta_h+\mu_h\right)\mu_m\big) + B_1\Big(\left(a+\eta_h+\mu_h\right)\left(\theta_2+\mu_h\right)\mu_m - m - B_3\beta_{hm}\Big(N_h\mu_h\big(\theta_2 -$$

$$\left(-1+\pi\right)\mu_h\big) + \theta_1\left(\theta_2+\mu_h\right)R_h^*\Big)\Big)\Big)^2\Big)\Big)\Big) / \left(2m + \left(B_2 - B_1\right)B_3N_h\beta_{hm}\beta_{mh}\mu_h\left(\theta_2+\mu_h\right)\right)$$

$$I_h^* = \frac{\mu_m k \beta_{mh} t}{k\beta_{mh} t m + \varphi \mu_m^2 (\mu_h + \eta_h + a)(\mu_h + \theta_2) R_e^2}$$

$$I_m^* = \Bigg(m + \frac{\varphi \mu_m^2 (\mu_h + \eta_h + a)(\mu_h + \theta_2) R_e^2}{k\beta_{mh} t} N_h^2 \beta_{mh} \mu_h \left(\pi B_2 \mu_h + B_1 \left(\theta_2 + (1-\pi) \mu_h \right) \right) +$$

$$N_h \beta_{mh} \left(\frac{k B_3 \beta_{hm} \beta_{mh} t}{\varphi(\mu_h + \theta_2) \mu_m R_e^2} \left(B_2 \mu_h + B_1 (\theta_2 + \mu_h) \right) + m + \frac{\varphi \mu_m^2 (\mu_h + \eta_h + a)(\mu_h + \theta_2) R_e^2}{k\beta_{mh} t} \right.$$

$$B_1 \theta_1 (\theta_2 + \mu_h) R_h^* \Big) + \sqrt{\Bigg(N_h^2 \beta_{mh}^2 \Bigg(4 B_1 B_2 \mu_h (a + \eta_h + \mu_h)(\theta_2 + \mu_h) \mu_m \left(\frac{k B_3 \beta_{hm} \beta_{mh} t}{\varphi(\mu_h + \theta_2) \mu_m R_e^2} \right. }$$

$$+ m + B_3 \beta_{hm} \left(N_h \mu_h + \theta_1 R_h^* \right) + \Big(B_2 \mu_h \left(-m\pi - B_3 N_h \beta_{hm} \mu_h + (a + \eta_h + \mu_h) \mu_m \right) +$$

$$B_1 \Big((a + \eta_h + \mu_h)(\theta_2 + \mu_h) \mu_m - m - B_3 \beta_{hm} \Big(N_h \mu_h \left(\theta_2 - (-1 + \pi) \mu_h \right) +$$

$$\theta_1 (\theta_2 + \mu_h) R_h^* \Big) \Big) \Big)^2 \Bigg) \Bigg) \Bigg) \Big/ \left(2 B_1 B_2 \beta_{mh}^2 \left((a + \eta_h + \mu_h) \mu_m + m \left(\sqrt{T} R_e + 1 \right) \left(\sqrt{T} R_e - 1 \right) \right) \right)$$

Where $T = \dfrac{\varphi \mu_m^2 (\mu_h + \eta_h + a)(\mu_h + \theta_2)\left(N_h \mu_h + \theta_1 R_h^* \right)}{m k \beta_{mh} t}$,

$$R_e^2 = \frac{-k B_3 \beta_{hm} \beta_{mh} t}{\varphi \mu_m (\mu_h + \theta_2) \mu_m (\mu_h + \eta_h + a)}$$

3.8.3 Local Stability of the Endemic Equilibrium

In order to analyse the stability of the endemic equilibrium, the additive compound matrices approach is used, using the idea of Lee and Lashari, (2014). Local stability of the endemic equilibrium point is determined by the variational matrix $\mathbf{J}\left(E^*\right)$ of the nonlinear system (3.34) corresponding to E^* as follows:

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - B_1\beta_{mh}\frac{I_m^*}{N_h}S_{h_1}^* - \mu_h S_{h_1}^* + \theta_1 R_h^* + \theta_2 S_{h_2}^* = G_1\left(S_{h_1}, S_{h_2}, I_h, I_m\right)$$
$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - B_2\beta_{mh}\frac{I_m^*}{N_h}S_{h_2}^* - \mu_h S_{h_2}^* - \theta_2 S_{h_2}^* = G_2\left(S_{h_1}, S_{h_2}, I_h, I_m\right)$$
$$\frac{dI_h}{dt} = \left(B_1 S_{h_1}^* + B_2 S_{h_2}^*\right)\beta_{mh}\frac{I_m^*}{N_h} - \left(\mu_h + \eta_h + a\right)I_h^* = G_3\left(S_{h_1}, S_{h_2}, I_h, I_m\right)$$
$$\frac{dI_m}{dt} = B_3\beta_{hm}\frac{I_h^*}{N_h}\left(mN_h - I_m^*\right) - \left(\mu_m\right)I_m^* = G_4\left(S_{h_1}, S_{h_2}, I_h, I_m\right) \qquad (3.36)$$

It follows that

$$\mathbf{J}\left(E^*\right) = \begin{pmatrix} \frac{\partial G_1}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_1}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_1}{\partial I_h}\left(E^*\right) & \frac{\partial G_1}{\partial I_m}\left(E^*\right) \\ \frac{\partial G_2}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_2}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_2}{\partial I_h}\left(E^*\right) & \frac{\partial G_2}{\partial I_m}\left(E^*\right) \\ \frac{\partial G_3}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_3}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_3}{\partial I_h}\left(E^*\right) & \frac{\partial G_3}{\partial I_m}\left(E^*\right) \\ \frac{\partial G_4}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_4}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_4}{\partial I_h}\left(E^*\right) & \frac{\partial G_4}{\partial I_m}\left(E^*\right) \end{pmatrix}$$

Hence the variational matrix of the non- linear model system (3.36) is obtained as

$$\mathbf{J}\left(E^*\right) = \begin{pmatrix} -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} & \theta_2 & 0 & -\frac{B_1\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} & 0 & -\frac{B_2\beta_{mh}S_{h_2}^*}{N_h} \\ \frac{B_1\beta_{mh}I_m^*}{N_h} & \frac{B_2\beta_{mh}I_m^*}{N_h} & -a - \eta_h - \mu_h & \frac{\beta_{mh}(B_1 S_{h_1}^* + B_2 S_{h_2}^*)}{N_h} \\ 0 & 0 & mB_3\beta_{hm} - \frac{B_3\beta_{hm}I_m^*}{N_h} & -\mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h} \end{pmatrix} \qquad (3.37)$$

The following lemma was stated and proved by McCluskey and Driessche (2004), to demonstrate the local stability of endemic equilibrium point E^*.

Lemma 3.2:

Let $\mathbf{J}\left(E^*\right)$ be a 4×4 real matrix. If $tr\left(\mathbf{J}\left(E^*\right)\right)$, $\det\left(\mathbf{J}\left(E^*\right)\right)$ and $\det\left(\mathbf{J}^{[2]}\left(E^*\right)\right)$ are all negative, then all eigenvalues of $\mathbf{J}\left(E^*\right)$ have negative real parts.

Using the above Lemma, we will study the stability of the endemic equilibrium.

Theorem 3.3: If $R_e > 1$, the endemic equilibrium E^* of the model (3.34) is locally asymptotically stable in Ω

Proof:

From the Jacobian matrix $\mathbf{J}\left(E^*\right)$ in (3.37), we have

$$tr\left(\mathbf{J}\left(E^*\right)\right) = -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} - \theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} - a - \eta_h - \mu_h$$

$$-\mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h} < 0$$

and

$$\det\left(\mathbf{J}\left(E^*\right)\right) = -\frac{1}{N_h^3}\left(\frac{kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m^2(\mu_h+\theta_2)R_e^2}\left(N_h\mu_m + B_3\beta_{hm}I_h^*\right)\left(N_h\mu_h + B_1\beta_{mh}I_m^*\right)\right.$$

$$\left(N_h\left(\theta_2+\mu_h\right)+B_2\beta_{mh}I_m^*\right)+B_3\beta_{hm}\beta_{mh}\mu_h\left(mN_h - I_m^*\right)\left(B_2N_h\left(\theta_2+\mu_h\right)S_{h_2}^*\right.$$

$$\left.\left.+B_1\left(N_h\left(\theta_2+\mu_h\right)S_{h_1}^* + B_2\beta_{mh}I_m^*\left(S_{h_1}^* + S_{h_2}^*\right)\right)\right)\right) < 0$$

Thus $\det\left(\mathbf{J}\left(E^*\right)\right) < 0$ if $R_e > 1$

where $(\mu_h + \eta_h + a) = \dfrac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m^2(\mu_h+\theta_2)R_e^2}$

Hence trace and determinant of the Jacobian matrix $\mathbf{J}\left(E^*\right)$ are all negative.

The second additive compound matrices are obtained from the following lemma.

Lemma 3.3:

To establish the second additive compound matrix $\left(\mathbf{J}^{[2]}\left(E^{*}\right)\right)$ of the Jacobian matrix $\mathbf{J}(E^{*})$, the following will be considered.

From the Jacobian matrix $\mathbf{J}\left(E^{*}\right)$, the second additive compound matrix $\left(\mathbf{J}^{[2]}\left(E^{*}\right)\right)$ is obtained by taking the coefficient of X from:

$$\begin{pmatrix} \det\mathbf{N}[1,2|1,2] & \det\mathbf{N}[1,2|1,3] & \det\mathbf{N}[1,2|1,4] & \det\mathbf{N}[1,2|2,3] & \det\mathbf{N}[1,2|2,4] & \det\mathbf{N}[1,2|3,4] \\ \det\mathbf{N}[1,3|1,2] & \det\mathbf{N}[1,3|1,3] & \det\mathbf{N}[1,3|1,4] & \det\mathbf{N}[1,3|2,3] & \det\mathbf{N}[1,3|2,4] & \det\mathbf{N}[1,3|3,4] \\ \det\mathbf{N}[1,4|1,2] & \det\mathbf{N}[1,4|1,3] & \det\mathbf{N}[1,4|1,4] & \det\mathbf{N}[1,4|2,3] & \det\mathbf{N}[1,4|2,4] & \det\mathbf{N}[1,4|3,4] \\ \det\mathbf{N}[2,3|1,2] & \det\mathbf{N}[2,3|1,3] & \det\mathbf{N}[2,3|1,4] & \det\mathbf{N}[2,3|2,3] & \det\mathbf{N}[2,3|2,4] & \det\mathbf{N}[2,3|3,4] \\ \det\mathbf{N}[2,4|1,2] & \det\mathbf{N}[2,4|1,3] & \det\mathbf{N}[2,4|1,4] & \det\mathbf{N}[2,4|2,3] & \det\mathbf{N}[2,4|2,4] & \det\mathbf{N}[2,4|3,4] \\ \det\mathbf{N}[3,4|1,2] & \det\mathbf{N}[3,4|1,3] & \det\mathbf{N}[3,4|1,4] & \det\mathbf{N}[3,4|2,3] & \det\mathbf{N}[3,4|2,4] & \det\mathbf{N}[3,4|3,4] \end{pmatrix}$$

, where $\mathrm{N}_{ij} = \left[\mathbf{J}\left(E^{*}\right) + \mathbf{I}X\right]$ and $\mathbf{I}$ is identity matrix. It follows that

$$\mathrm{N} = \left[\mathbf{J}\left(E^{*}\right) + \mathbf{I}X\right] =$$

$$\begin{bmatrix} -\mu_h - \dfrac{B_1\beta_{mh}I_m^*}{N_h} & \theta_2 & 0 & -\dfrac{B_1\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & -\theta_2 - \mu_h - \dfrac{B_2\beta_{mh}I_m^*}{N_h} & 0 & -\dfrac{B_2\beta_{mh}S_{h_2}^*}{N_h} \\ \dfrac{B_1\beta_{mh}I_m^*}{N_h} & \dfrac{B_2\beta_{mh}I_m^*}{N_h} & -a - \eta_h - \mu_h & \dfrac{\beta_{mh}(B_1S_{h_1}^* + B_2S_{h_2}^*)}{N_h} \\ 0 & 0 & mB_3\beta_{hm} - \dfrac{B_3\beta_{hm}I_m^*}{N_h} & -\mu_m - \dfrac{B_3\beta_{hm}I_h^*}{N_h} \end{bmatrix} +$$

$$\begin{bmatrix} X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X \end{bmatrix}$$

This is equivalent to

$$\mathbf{N}=\begin{bmatrix} -\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}+X & \theta_2 & 0 & -\frac{B_1\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & -\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}+X & 0 & -\frac{B_2\beta_{mh}S_{h_2}^*}{N_h} \\ \frac{B_1\beta_{mh}I_m^*}{N_h} & \frac{B_2\beta_{mh}I_m^*}{N_h} & -a-\eta_h-\mu_h+X & \frac{\beta_{mh}(B_1S_{h_1}^*+B_2S_{h_2}^*)}{N_h} \\ 0 & 0 & mB_3\beta_{hm}-\frac{B_3\beta_{hm}I_m^*}{N_h} & -\mu_m-\frac{B_3\beta_{hm}I_h^*}{N_h}+X \end{bmatrix}$$

Then $\det\mathbf{N}[1,2\,|\,1,2]=\begin{vmatrix} -\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}+X & \theta_2 \\ 0 & -\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}+X \end{vmatrix}$

Or $\det\mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}+X\right)\left(-\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}+X\right)-0$ then yield

$$\det\mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}+X\right)\left(-\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}+X\right)$$

Consequently

$$\det\mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}\right)\left(-\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}\right)+$$

$$\left(-\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}\right)X+\left(-\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}\right)X+X^2$$

Take coefficient of X and obtain

$$\det\mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}\right)+\left(-\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}\right)$$

Therefore $\det\mathbf{N}[1,2\,|\,1,2]=-\mu_h-\frac{B_1\beta_{mh}I_m^*}{N_h}-\theta_2-\mu_h-\frac{B_2\beta_{mh}I_m^*}{N_h}$ then

$$\det \mathbf{N}[1,2|1,3] = \begin{vmatrix} -\mu_h - \frac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h} + X & 0 \\ 0 & 0 \end{vmatrix} \text{ or}$$

$$\det \mathbf{N}[1,2|1,3] = \left(-\mu_h + \frac{B_1(-1+u_2)\beta_{mh}I_m^*}{N_h} + X \right) 0 - 0$$

Therefore $\det \mathbf{N}[1,2|1,3] = 0$

Other determinants will be calculated in the same way to obtain the following matrix

$$\mathbf{J}_{(E^*)}^{[2]} = \begin{bmatrix} a_{11} & 0 & -\frac{B_2\beta_{mh}S_{h_2}^*}{N_h} & 0 & \frac{B_1\beta_{mh}S_{h_1}^*}{N_h} & 0 \\ \frac{B_2\beta_{mh}I_m^*}{N_h} & a_{22} & a_{23} & \theta_2 & 0 & \frac{B_1\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & mB_3\beta_{hm} - \frac{B_3\beta_{hm}I_m^*}{N_h} & a_{33} & 0 & \theta_2 & 0 \\ -\frac{B_1\beta_{mh}I_m^*}{N_h} & 0 & 0 & a_{44} & a_{45} & \frac{B_2\beta_{mh}S_{h_2}^*}{N_h} \\ 0 & 0 & 0 & mB_3\beta_{hm} - \frac{B_3\beta_{hm}I_m^*}{N_h} & a_{55} & 0 \\ 0 & 0 & \frac{B_1\beta_{mh}I_m^*}{N_h} & 0 & \frac{B_2\beta_{mh}I_m^*}{N_h} & a_{66} \end{bmatrix}$$

Where $a_{11} = -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} - \theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h}$

$$a_{22} = -\frac{B_1\beta_{mh}I_m^*}{N_h} - a - \eta_h - 2\mu_h \quad a_{23} = \frac{\beta_{mh}\left(B_1S_{h_1}^* + B_2S_{h_2}^*\right)}{N_h}$$

$$a_{33} = -\mu_h - \frac{B_1\beta_{mh}I_m^*}{N_h} - \mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h}$$

$$a_{44} = -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} - a - \eta_h - \mu_h, \quad a_{45} = \frac{\beta_{mh}\left(B_1S_{h_1}^* + B_2S_{h_2}^*\right)}{N_h}$$

$$a_{55} = -\theta_2 - \mu_h - \frac{B_2\beta_{mh}I_m^*}{N_h} - \mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h}, \quad a_{66} = -a - \eta_h - \mu_h - \mu_m - \frac{B_3\beta_{hm}I_h^*}{N_h}$$

Therefore, it gives

$$\det(\mathbf{J}^{[2]}(E^*)) = -\frac{1}{N_h^6}(B_3\beta_{hm}\beta_{mh}\left(mN_h - I_m^*\right)\Big(fR_0^2\left(mN_h - I_m^*\right)F + \left(N_h\left(a+\eta_h+2\mu_h\right)+ B_1\beta_{mh}I_m^*\right)(B_2\beta_{mh}I_m^*V + \left(N_h\left(a+\eta_h+\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*\right)\Big(\Big(N_h\left(\mu_h+\mu_m\right) + B_3\beta_{hm}I_h^* + B_1\beta_{mh}I_m^*\Big)\Big(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*\Big)\Big(B_1S_{h_1}^* + B_2S_{h_2}^*\Big)+ \Big(\beta_{mh}B_1I_m^*B_1\left(N_h\left(\mu_h+\mu_m\right)+B_3\beta_1I_h^*\right)S_{h_1}^*\left(\sqrt{\delta}R_e+1\right)\left(\sqrt{\delta}R_e-1\right)+\beta_{mh}B_1I_m^* B_2N_h\theta_2S_{h_2}^*\Big)\Big)\Big)\Big)+\Big(N_h\left(\theta_2+\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*+B_2\beta_{mh}I_m^*\Big)\Big(G+N_h\big(a+\eta_h+ \mu_h+\mu_m\big)\left(\sqrt{y}R_e+1\right)\left(\sqrt{y}R_e-1\right)\Big(fR_0^2B_1B_2N_h\beta_{mh}\theta_2I_m^*\left(mN_h-I_m^*\right)S_{h_2}^* + \left(N_h\left(a+\eta_h+\theta_2+2\mu_h\right)+B_2\beta_{mh}I_m^*\right)\Big(\Big(N_h\left(a+\eta_h+2\mu_h\right)+B_1\beta_{mh}I_m^*\Big)Q +\Big(B_2^2B_3\beta_{hm}\beta_{mh}\left(mN_h-I_m^*\right)\beta_{mh}I_m^*S_{h_2}^* + fR_e^2\left(mN_h-I_m^*\right)J\Big)\Big)\Big)\Big)\Big) < 0$$ where

$$f = \frac{(\mu_h+\eta_h+a)\varphi\mu_m(\mu_h+\theta_2)\mu_m}{kt}, \quad \delta = \frac{B_1I_m^*S_{h_1}^*f}{\left(N_h\left(\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*\right)S_{h_1}^*B_3kt\beta_{hm}}$$

$$F = (B_1\left(N_h\left(a+\eta_h+\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*\right)S_{h_1}^* + B_1^2\beta_{mh}I_m^*S_{h_1}^* + B_2\left(N_h\left(a+\eta_h+\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*+B_2\beta_{mh}I_m^*\right)S_{h_2}^*\Big) \left(B_2N_h\left(\theta_2+2\mu_h\right)S_{h_2}^* + B_1(N_h\left(\theta_2+2\mu_h\right)S_{h_1}^* + B_2\beta_{mh}I_m^*\left(S_{h_1}^*+S_{h_2}^*\right)\right)\Big)$$

$$G = B_1B_3\beta_{hm}\beta_{mh}^2\left(mN_h-I_m^*\right)I_m^*\left(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*\right) \left(B_1\left(N_h\left(a+\eta_h+\theta_2+2\mu_h\right)+B_2\beta_{mh}I_m^*\right)S_{h_1}^* + B_2N_h\theta_2S_{h_2}^*\right)$$

$$J = \left(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*\right)\left(B_1S_{h_1}^*+B_2S_{h_2}^*\right)$$

$$V = \left(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*\right)\Big(B_2\left(N_h\left(\mu_h+\mu_m\right)+\right.$$

$$B_3\beta_{hm}I_h^*\Big)+B_1\left(N_h\theta_2+B_2\beta_{mh}I_m^*\right)\Big)S_{h_2}^*$$

$$Q=\left(N_h\left(\mu_h+\mu_m\right)+B_3\beta_{hm}I_h^*+B_1\beta_{mh}I_m^*\right)(N_h\left(\theta_2+2\mu_h\right)+\left(B_1+B_2\right)\beta_{mh}I_m^*)$$

$$y=\frac{fI_h^*}{N_h\left(a+\eta_h+\mu_h+\mu_m\right)kt\beta_{mh}},\quad R_e^2=\frac{-kB_3\beta_{hm}\beta_{mh}t}{\varphi\mu_m\left(\mu_h+\theta_2\right)\mu_m\left(\mu_h+\eta_h+a\right)}$$

Thus, from the lemma 3.2, the endemic equilibrium E^* of the model system (3.34) is locally asymptotically stable in Ω.

3.8.4 Global Stability of Endemic Equilibrium Point (EEP)

Theorem 3.4:

If $R_e>1$ the endemic equilibrium E^* of the model system (3.1) is globally asymptotically stable

Proof: To establish the global stability of endemic equilibrium E^* we construct the following positive Lyapunov function V as follows:

$$V\left(S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*\ \&\ \mathrm{I}_m^*\right)=\left(S_{h_1}-S_{h_1}^*\ln S_{h_1}\right)+\left(S_{h_2}-S_{h_2}^*\ln S_{h_2}\right)+\left(I_h-I_h^*\ln I_h\right)+$$

$$\left(T_h-T_h^*\ln T_h\right)+\left(R_h-R_h^*\ln R_h\right)+\left(A_m-A_m^*\ln A_m\right)+\left(S_m-S_m^*\ln S_m\right)+\left(I_m-I_m^*\ln I_m\right)\quad(3.38)$$

Direct calculation of the derivative of V along the solutions of (3.38) gives,

$$\frac{dV}{dt}\left(S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*\ \&\ \mathrm{I}_m^*\right)=\left(1-\frac{S_{h_1}^*}{S_{h_1}}\right)\frac{dS_{h_1}}{dt}+\left(1-\frac{S_{h_2}^*}{S_{h_2}}\right)\frac{dS_{h_2}}{dt}+\left(1-\frac{I_h^*}{I_h}\right)\frac{dI_h}{I_h}+$$

$$\left(1-\frac{T_h^*}{T_h}\right)\frac{dT_h}{dt}+\left(1-\frac{R_h^*}{R_h}\right)\frac{dR_h}{dt}+\left(1-\frac{A_m^*}{A_m}\right)\frac{dA_m}{dt}+\left(1-\frac{S_m^*}{S_m}\right)\frac{dS_m}{S_m}+\left(1-\frac{I_m^*}{I_m}\right)\frac{dI_m}{dt}$$

Consequently

$$\frac{dV}{dt}=\left(\frac{S_{h_1}-S_{h_1}^*}{S_{h_1}}\right)\frac{dS_{h_1}}{dt}+\left(\frac{S_{h_2}-S_{h_2}^*}{S_{h_2}}\right)\frac{dS_{h_2}}{dt}+\left(\frac{I_h-I_h^*}{I_h}\right)\frac{dI_h}{dt}+\left(\frac{T_h-T_h^*}{T_h}\right)\frac{dT_h}{dt}+$$

$$\left(\frac{R_h-R_h^*}{R_h}\right)\frac{dR_h}{dt}+\left(\frac{A_m-A_m^*}{A_m}\right)\frac{dA_m}{dt}+\left(\frac{S_m-S_m^*}{S_m}\right)\frac{dS_m}{dt}+\left(\frac{I_m-I_m^*}{I_m}\right)\frac{dI_m}{dt}$$

Substituting $S_{h_1}=S_{h_1}-S_{h_1}^*$, $S_{h_2}=S_{h_2}-S_{h_2}^*$, $I_h=I_h-I_h^*$, $T_h=T_h-T_h^*$,

$R_h=R_h-R_h^*, A_m=A_m-A_m^*$, $S_m=S_m-S_m^*$ and $I_m=I_m-I_m^*$ into (3.1) gives

$$\frac{dS_{h_1}}{dt}=(1-\pi)\mu_h N_h-\left(B_1\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}+\mu_h\right)\left(S_{h_1}-S_{h_1}^*\right)+\theta_1\left(R_h-R_h^*\right)+\theta_2\left(S_{h_2}-S_{h_2}^*\right)$$

$$\frac{dS_{h_2}}{dt}=\pi\mu_h N_h-\left(B_2\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}+\mu_h+\theta_2\right)\left(S_{h_2}-S_{h_2}^*\right)$$

$$\frac{dI_h}{dt}=\left(B_1\left(S_{h_1}-S_{h_1}^*\right)+B_2\left(S_{h_2}-S_{h_2}^*\right)\right)\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}-\left(\mu_h+\eta_h+a\right)\left(I_h-I_h^*\right)$$

$$\frac{dT_h}{dt}=\eta_h\left(I_h-I_h^*\right)-\left(\mu_h+\delta_h\right)\left(T_h-T_h^*\right)$$

$$\frac{dR_h}{dt}=\delta_h\left(T_h-T_h^*\right)-\left(\mu_h+\theta_1\right)\left(R_h-R_h^*\right)$$

$$\frac{dA_m}{dt}=\varphi\left(1-\frac{\left(A_m-A_m^*\right)}{kN_h}\right)\left(\left(S_m-S_m^*\right)+\left(I_m-I_m^*\right)\right)-\left(\mu_A+\eta_A\right)\left(A_m-A_m^*\right)$$

$$\frac{dS_m}{dt}=\eta_A\left(A_m-A_m^*\right)-\left(B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}+\mu_m\right)\left(S_m-S_m^*\right)$$

$$\frac{dI_m}{dt}=B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}\left(S_m-S_m^*\right)-\mu_m\left(I_m-I_m^*\right)$$

It follows that

$$\frac{dV}{dt}=f_1+f_2+f_3+f_4+f_5+f_6+f_7+f_8 \tag{3.39}$$

Where,

$$f_1=\left(\frac{S_{h_1}-S_{h_1}^*}{S_{h_1}}\right)\left\{(1-\pi)\mu_h N_h-\left(B_1\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}+\mu_h\right)\left(S_{h_1}-S_{h_1}^*\right)+\theta_1\left(R_h-R_h^*\right)+\theta_2\left(S_{h_2}-S_{h_2}^*\right)\right\} \quad (3.40)$$

$$f_2=\left(\frac{S_{h_2}-S_{h_2}^*}{S_{h_2}}\right)\left\{\pi\mu_h N_h-\left(B_2\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}+\mu_h+\theta_2\right)\left(S_{h_2}-S_{h_2}^*\right)\right\} \quad (3.41)$$

$$f_3=\left(\frac{I_h-I_h^*}{I_h}\right)\left\{\left(B_1\left(S_{h_1}-S_{h_1}^*\right)+B_2\left(S_{h_2}-S_{h_2}^*\right)\right)\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}-\left(\mu_h+\eta_h+a\right)\left(I_h-I_h^*\right)\right\} \quad (3.42)$$

$$f_4=\left(\frac{T_h-T_h^*}{T_h}\right)\left\{\eta_h\left(I_h-I_h^*\right)-\left(\mu_h+\delta_h\right)\left(T_h-T_h^*\right)\right\} \quad (3.43)$$

$$f_5=\left(\frac{R_h-R_h^*}{R_h}\right)\left\{\delta_h\left(T_h-T_h^*\right)-\left(\mu_h+\theta_1\right)\left(R_h-R_h^*\right)\right\} \quad (3.44)$$

$$f_6=\left(\frac{A_m-A_m^*}{A_m}\right)\left\{\varphi\left(1-\frac{\left(A_m-A_m^*\right)}{kN_h}\right)\left(\left(S_m-S_m^*\right)+\left(I_m-I_m^*\right)\right)-\left(\mu_A+\eta_A\right)\left(A_m-A_m^*\right)\right\} \quad (3.45)$$

$$f_7=\left(\frac{S_m-S_m^*}{S_m}\right)\left\{\eta_A\left(A_m-A_m^*\right)-\left(B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}+\mu_m\right)\left(S_m-S_m^*\right)\right\} \quad (3.46)$$

$$f_8=\left(\frac{I_m-I_m^*}{I_m}\right)\left\{B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}\left(S_m-S_m^*\right)-\mu_m\left(I_m-I_m^*\right)\right\} \quad (3.47)$$

We simplify (3.40) to (3.47) as follows

$$f_1=\left(\frac{S_{h_1}-S_{h_1}^*}{S_{h_1}}\right)\left\{(1-\pi)\mu_h N_h-\left(B_1\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}+\mu_h\right)\left(S_{h_1}-S_{h_1}^*\right)+\theta_1\left(R_h-R_h^*\right)+\theta_2\left(S_{h_2}-S_{h_2}^*\right)\right\}$$

or

$$f_1 = \mu_h N_h \left(\frac{S_{h_1} - S^*_{h_1}}{S_{h_1}} \right) - \pi\mu_h N_h \left(\frac{S_{h_1} - S^*_{h_1}}{S_{h_1}} \right) - \left(\frac{B_1\beta_{mh} I_m}{N_h} - \frac{B_1\beta_{mh} I^*_m}{N_h} + \mu_h \right) \left(S_{h_1} - S^*_{h_1} \right) \left(\frac{S_{h_1} - S^*_{h_1}}{S_{h_1}} \right) +$$
$$\left(\theta_1 R_h - \theta_1 R^*_h \right) \left(\frac{S_{h_1} - S^*_{h_1}}{S_{h_1}} \right) + \left(\theta_2 S_{h_2} - \theta_2 S^*_{h_2} \right) \left(\frac{S_{h_1} - S^*_{h_1}}{S_{h_1}} \right). \text{Then}$$

$$f_1 = \mu_h N_h \left(1 - \frac{S^*_{h_1}}{S_{h_1}} \right) - \pi\mu_h N_h \left(1 - \frac{S^*_{h_1}}{S_{h_1}} \right) - \left(\frac{B_1\beta_{mh} I_m}{N_h} - \frac{B_1\beta_{mh} I^*_m}{N_h} + \mu_h \right) \left(S_{h_1} - S^*_{h_1} \right) \left(\frac{S_{h_1} - S^*_{h_1}}{S_{h_1}} \right) +$$
$$\left(\theta_1 R_h - \theta_1 R^*_h \right) \left(1 - \frac{S^*_{h_1}}{S_{h_1}} \right) + \left(\theta_2 S_{h_2} - \theta_2 S^*_{h_2} \right) \left(1 - \frac{S^*_{h_1}}{S_{h_1}} \right). \text{ Hence}$$

$$f_1 = \mu_h N_h + \pi\mu_h N_h \frac{S^*_{h_1}}{S_{h_1}} + \frac{B_1\beta_{mh} I^*_m}{N_h} \frac{\left(S_{h_1} - S^*_{h_1} \right)^2}{S_{h_1}} + \theta_1 R_h + \theta_1 R^*_h \frac{S^*_{h_1}}{S_{h_1}} + \theta_2 S_{h_2} + \theta_2 S^*_{h_2} \frac{S^*_{h_1}}{S_{h_1}}$$

$$-\mu_h N_h \frac{S^*_{h_1}}{S_{h_1}} - \pi\mu_h N_h - \frac{B_1\beta_{mh} I_m}{N_h} \frac{\left(S_{h_1} - S^*_{h_1} \right)^2}{S_{h_1}} - \mu_h \frac{\left(S_{h_1} - S^*_{h_1} \right)^2}{S_{h_1}} - \theta_1 R^*_h - \theta_1 R_h \frac{S^*_{h_1}}{S_{h_1}}$$

$$-\theta_2 S^*_{h_2} - \theta_2 S_{h_2} \frac{S^*_{h_1}}{S_{h_1}}$$

Then for the equation (3.41) we have

$$f_2 = \left\{ \left(\pi\mu_h N_h - \pi\mu_h N_h \frac{S^*_{h_2}}{S_{h_2}} \right) - \left(\frac{B_2\beta_{mh} I_m}{N_h} - \frac{B_2\beta_{mh} I^*_m}{N_h} + \mu_h + \theta_2 \right) \frac{\left(S_{h_2} - S^*_{h_2} \right)^2}{S_{h_2}} \right\}$$

This yields

$$f_2 = \pi\mu_h N_h + \frac{B_2\beta_{mh} I^*_m}{N_h} \frac{\left(S_{h_2} - S^*_{h_2} \right)^2}{S_{h_2}} - \pi\mu_h N_h \frac{S^*_{h_2}}{S_{h_2}}$$

$$-\frac{B_2\beta_{mh} I_m}{N_h} \frac{\left(S_{h_2} - S^*_{h_2} \right)^2}{S_{h_2}} - \mu_h \frac{\left(S_{h_2} - S^*_{h_2} \right)^2}{S_{h_2}} - \theta_2 \frac{\left(S_{h_2} - S^*_{h_2} \right)^2}{S_{h_2}}$$

For the equation (3.42) we have

$$f_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left(B_1\left(S_{h_1} - S_{h_1}^*\right) + B_2\left(S_{h_2} - S_{h_2}^*\right)\right)\beta_{mh}\frac{\left(I_m - I_m^*\right)}{N_h} - \left(\mu_h + \eta_h + a\right)\left(I_h - I_h^*\right)\right\}$$

or

$$f_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left(\left(B_1 S_{h_1} - B_1 S_{h_1}^*\right) + \left(B_2 S_{h_2} - B_2 S_{h_2}^*\right)\right)\frac{\left(\beta_{mh} I_m - \beta_{mh} I_m^*\right)}{N_h} - \left(\mu_h + \eta_h + a\right)\left(I_h - I_h^*\right)\right\}.$$

Then

$$f_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left(\left(B_1 S_{h_1} - B_1 S_{h_1}^*\right) + \left(B_2 S_{h_2} - B_2 S_{h_2}^*\right)\right)\left(\frac{\beta_{mh} I_m}{N_h} - \frac{\beta_{mh} I_m^*}{N_h}\right) - \left(\mu_h + \eta_h + a\right)\left(I_h - I_h^*\right)\right\}$$

or

$$f_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\begin{array}{l}\left(\left(B_1 S_{h_1} - B_1 S_{h_1}^*\right)\left(\frac{\beta_{mh} I_m}{N_h} - \frac{\beta_{mh} I_m^*}{N_h}\right) + \left(B_2 S_{h_2} - B_2 S_{h_2}^*\right)\left(\frac{\beta_{mh} I_m}{N_h} - \frac{\beta_{mh} I_m^*}{N_h}\right)\right) \\ -\left(\mu_h + \eta_h + a\right)\left(I_h - I_h^*\right)\end{array}\right\}.$$

It follows that

$$f_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left(\left(\frac{\beta_{mh} I_m}{N_h}\left(B_1 S_{h_1} - B_1 S_{h_1}^*\right) - \frac{\beta_{mh} I_m^*}{N_h}\left(B_1 S_{h_1} - B_1 S_{h_1}^*\right)\right) + \left(\frac{\beta_{mh} I_m}{N_h}\left(B_2 S_{h_2} - B_2 S_{h_2}^*\right)\right.\right.\right.$$
$$\left.\left.\left. - \frac{\beta_{mh} I_m^*}{N_h}\left(B_2 S_{h_2} - B_2 S_{h_2}^*\right)\right)\right) - \left(\mu_h + \eta_h + a\right)\left(I_h - I_h^*\right)\right\}$$

or

$$f_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left(\frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1} - \frac{\beta_{mh} I_m}{N_h} B_1 S_{h_1}^* - \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1} + \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1}^* + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}\right.\right.$$
$$\left.\left. - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* - \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2} + \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}^*\right) - \left(\mu_h + \eta_h + a\right)\left(I_h - I_h^*\right)\right\}$$

Therefore

$$f_3=\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}+\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}^*\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}^*+\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}+$$

$$\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}^*\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}^*-\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}\frac{I_h^*}{I_h}-\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}^*$$

$$-\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}-\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}^*\frac{I_h^*}{I_h}-\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}\frac{I_h^*}{I_h}-\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}^*-\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}$$

$$-\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}^*\frac{I_h^*}{I_h}-\left(\mu_h+\eta_h+a\right)\frac{\left(I_h-I_h^*\right)^2}{I_h}$$

Equation (3.43) becomes

$$f_4=\left(\frac{T_h-T_h^*}{T_h}\right)\left\{\eta_h\left(I_h-I_h^*\right)-\left(\mu_h+\delta_h\right)\left(T_h-T_h^*\right)\right\}\quad \text{or}$$

$$f_4=\left(\eta_hI_h-\eta_hI_h^*\right)\left(\frac{T_h-T_h^*}{T_h}\right)-\left(\mu_h+\delta_h\right)\left(T_h-T_h^*\right)\left(\frac{T_h-T_h^*}{T_h}\right).\text{ Then}$$

$$f_4=\left(\eta_hI_h-\eta_hI_h^*\right)\left(1-\frac{T_h^*}{T_h}\right)-\left(\mu_h+\delta_h\right)\frac{\left(T_h-T_h^*\right)^2}{T_h}.\text{ Hence}$$

$$f_4=\eta_hI_h+\eta_hI_h^*\frac{T_h^*}{T_h}-\eta_hI_h^*-\eta_hI_h\frac{T_h^*}{T_h}-\left(\mu_h+\delta_h\right)\frac{\left(T_h-T_h^*\right)^2}{T_h}$$

For the equation (3.44) we have

$$f_5=\left\{\delta_hT_h\left(\frac{R_h-R_h^*}{R_h}\right)-\delta_hT_h^*\left(\frac{R_h-R_h^*}{R_h}\right)-\left(\mu_h+\theta_1\right)\left(R_h-R_h^*\right)\left(\frac{R_h-R_h^*}{R_h}\right)\right\}.$$

Consequently

$$f_5=\delta_hT_h+\delta_hT_h^*\frac{R_h^*}{R_h}-\delta_hT_h\frac{R_h^*}{R_h}-\delta_hT_h^*-\left(\mu_h+\theta_1\right)\frac{\left(R_h-R_h^*\right)^2}{R_h}.$$

Then equation (3.45) becomes

$$f_6 = \left(\frac{A_m - A_m^*}{A_m}\right)\left\{\left(\left(\varphi\left(S_m - S_m^*\right) - \frac{\varphi\left(A_m - A_m^*\right)}{kN_h}\left(S_m - S_m^*\right)\right) + \left(\varphi\left(I_m - I_m^*\right) - \frac{\varphi\left(A_m - A_m^*\right)}{kN_h}\left(I_m - I_m^*\right)\right)\right)\right.$$

$$\left. -\left(\mu_A + \eta_A\right)\left(A_m - A_m^*\right)\right\}$$

or

$$f_6 = \left(\frac{A_m - A_m^*}{A_m}\right)\left\{\left(\left(\varphi S_m - \varphi S_m^* - \frac{\varphi\left(A_m - A_m^*\right)}{kN_h}S_m + \frac{\varphi\left(A_m - A_m^*\right)}{kN_h}S_m^*\right) + \right.\right.$$

$$\left.\left.\left(\varphi I_m - \varphi I_m^* - \frac{\varphi\left(A_m - A_m^*\right)}{kN_h}I_m + \frac{\varphi\left(A_m - A_m^*\right)}{kN_h}I_m^*\right)\right) - \left(\mu_A + \eta_A\right)\left(A_m - A_m^*\right)\right\}.$$

Consequently

$$f_6 = \varphi S_m + \varphi S_m^* \frac{A_m^*}{A_m} + \varphi I_m + \frac{\varphi\left(A_m - A_m^*\right)^2}{kN_h A_m} S_m^* + \varphi I_m^* \frac{A_m^*}{A_m} + \frac{\varphi\left(A_m - A_m^*\right)^2}{kN_h A_m} I_m^* - \varphi S_m \frac{A_m^*}{A_m} -$$

$$\varphi S_m^* - \frac{\varphi\left(A_m - A_m^*\right)^2}{kN_h A_m} S_m - \varphi I_m \frac{A_m^*}{A_m} - \varphi I_m^* - \frac{\varphi\left(A_m - A_m^*\right)^2}{kN_h A_m} I_m - \left(\mu_A + \eta_A\right)\frac{\left(A_m - A_m^*\right)^2}{A_m}$$

For equation (3.46) we have

$$f_7 = \left(\frac{S_m - S_m^*}{S_m}\right)\left\{\eta_A A_m - \eta_A A_m^* - \left(\frac{B_3\beta_{hm} I_h}{N_h} - \frac{B_3\beta_{hm} I_h^*}{N_h} + \mu_m\right)\left(S_m - S_m^*\right)\right\}.$$ Then

$$f_7 = \eta_A A_m + \eta_A A_m^* \frac{S_m^*}{S_m} + \frac{B_3\beta_{hm} I_h^*}{N_h}\frac{\left(S_m - S_m^*\right)^2}{S_m} - \eta_A A_m \frac{S_m^*}{S_m} - \eta_A A_m^* - \frac{B_3\beta_{hm} I_h}{N_h}\frac{\left(S_m - S_m^*\right)^2}{S_m} - \mu_m \frac{\left(S_m - S_m^*\right)^2}{S_m}$$

For equation (3.47) we have

$$f_8 = \left(\frac{I_m - I_m^*}{I_m}\right)\left\{\left(\frac{B_3\beta_{hm} I_h}{N_h} - \frac{B_3\beta_{hm} I_h^*}{N_h}\right)\left(S_m - S_m^*\right) - \mu_m\left(I_m - I_m^*\right)\right\}$$ or

$$f_8 = \left(\frac{I_m - I_m^*}{I_m}\right)\left\{\left(\frac{B_3\beta_{hm} I_h}{N_h} S_m - \frac{B_3\beta_{hm} I_h}{N_h} S_m^* - \frac{B_3\beta_{hm} I_h^*}{N_h} S_m + \frac{B_3\beta_{hm} I_h^*}{N_h} S_m^*\right) - \mu_m\left(I_m - I_m^*\right)\right\}$$

Therefore

$$f_8 = \frac{B_3\beta_{hm}I_h}{N_h}S_m + \frac{B_3\beta_{hm}I_h^*}{N_h}S_m\frac{I_m^*}{I_m} + \frac{B_3\beta_{hm}I_h^*}{N_h}S_m^* + \frac{B_3\beta_{hm}I_h}{N_h}S_m^*\frac{I_m^*}{I_m}$$

$$-\frac{B_3\beta_{hm}I_h}{N_h}S_m\frac{I_m^*}{I_m} - \frac{B_3\beta_{hm}I_h}{N_h}S_m^* - \frac{B_3\beta_{hm}I_h^*}{N_h}S_m - \frac{B_3\beta_{hm}I_h^*}{N_h}S_m^*\frac{I_m^*}{I_m} - \mu_m\frac{\left(I_m - I_m^*\right)^2}{I_m}$$

Collect positive and negative terms together in the system (3.39),we obtain

$$\frac{dV}{dt} = A - B \tag{3.48}$$

where

$$A = \mu_h N_h + \pi\mu_h N_h\frac{S_{h_1}^*}{S_{h_1}} + \frac{B_1\beta_{mh}I_m^*}{N_h}\frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} + \theta_1 R_h + \theta_1 R_h^*\frac{S_{h_1}^*}{S_{h_1}} + \theta_2 S_{h_2} + \theta_2 S_{h_2}^*\frac{S_{h_1}^*}{S_{h_1}} + \pi\mu_h N_h +$$

$$\frac{B_2\beta_{mh}I_m^*}{N_h}\frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} + \frac{\beta_{mh}I_m}{N_h}B_1S_{h_1} + \frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}^*\frac{I_h^*}{I_h} + \frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}\frac{I_h^*}{I_h} + \frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}^* +$$

$$\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2} + \frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}^*\frac{I_h^*}{I_h} + \frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}\frac{I_h^*}{I_h} + \frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}^* + \eta_h I_h + \eta_h I_h^*\frac{T_h^*}{T_h} + \delta_h T_h +$$

$$\delta_h T_h^*\frac{R_h^*}{R_h} + \varphi S_m + \varphi S_m^*\frac{A_m^*}{A_m} + \varphi I_m + \frac{\varphi\left(A_m - A_m^*\right)^2}{kN_hA_m}S_m^* + \varphi I_m^*\frac{A_m^*}{A_m} + \frac{\varphi\left(A_m - A_m^*\right)^2}{kN_hA_m}I_m^* + \eta_A A_m$$

$$+\eta_A A_m^*\frac{S_m^*}{S_m} + \frac{B_3\beta_{hm}I_h^*}{N_h}\frac{\left(S_m - S_m^*\right)^2}{S_m} + \frac{B_3\beta_{hm}I_h}{N_h}S_m + \frac{B_3\beta_{hm}I_h^*}{N_h}S_m\frac{I_m^*}{I_m} + \frac{B_3\beta_{hm}I_h^*}{N_h}S_m^* + \frac{B_3\beta_{hm}I_h}{N_h}S_m^*\frac{I_m^*}{I_m}$$

$$B = -\mu_h N_h\frac{S_{h_1}^*}{S_{h_1}} - \pi\mu_h N_h - \frac{B_1\beta_{mh}I_m}{N_h}\frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \mu_h\frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \theta_1 R_h^* - \theta_1 R_h\frac{S_{h_1}^*}{S_{h_1}} - \theta_2 S_{h_2}^* -$$

$$\theta_2 S_{h_2}\frac{S_{h_1}^*}{S_{h_1}} - \pi\mu_h N_h\frac{S_{h_2}^*}{S_{h_2}} - \frac{B_2\beta_{mh}I_m}{N_h}\frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - \mu_h\frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - \theta_2\frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - \frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}\frac{I_h^*}{I_h}$$

$$-\frac{\beta_{mh}I_m}{N_h}B_1S_{h_1}^* - \frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1} - \frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}^*\frac{I_h^*}{I_h} - \frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}\frac{I_h^*}{I_h} - \frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}^* - \frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2} -$$

$$\frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}^* \frac{I_h^*}{I_h} - (\mu_h + \eta_h + a) \frac{(I_h - I_h^*)^2}{I_h} - \eta_h I_h^* - \eta_h I_h \frac{T_h^*}{T_h} - (\mu_h + \delta_h) \frac{(T_h - T_h^*)^2}{T_h} - \delta_h T_h \frac{R_h^*}{R_h}$$
$$-\delta_h T_h^* - (\mu_h + \theta_1) \frac{(R_h - R_h^*)^2}{R_h} - \varphi S_m \frac{A_m^*}{A_m} - \varphi S_m^* - \frac{\varphi (A_m - A_m^*)^2}{k N_h A_m} S_m - \varphi I_m \frac{A_m^*}{A_m} - \varphi I_m^* -$$

$$\frac{\varphi (A_m - A_m^*)^2}{k N_h A_m} I_m - (\mu_A + \eta_A) \frac{(A_m - A_m^*)^2}{A_m} - \eta_A A_m \frac{S_m^*}{S_m} - \eta_A A_m^* - \frac{B_3 \beta_{hm} I_h}{N_h} \frac{(S_m - S_m^*)^2}{S_m} -$$

$$\mu_m \frac{(S_m - S_m^*)^2}{S_m} - \frac{B_3 \beta_{hm} I_h}{N_h} S_m \frac{I_m^*}{I_m} - \frac{B_3 \beta_{hm} I_h}{N_h} S_m^* - \frac{B_3 \beta_{hm} I_h^*}{N_h} S_m - \frac{B_3 \beta_{hm} I_h^*}{N_h} S_m^* \frac{I_m^*}{I_m} - \mu_m \frac{(I_m - I_m^*)^2}{I_m}$$

Thus from equation (3.48), if $A < \mathrm{B}$ then $\frac{dV}{dt}$ will be negative definite, meaning that $\frac{dV}{dt} < 0$. It follows that $\frac{dV}{dt} = 0$ if and only if $S_{h_1} = S_{h_1}^*$, $S_{h_2} = S_{h_2}^*$, $I_h = I_h^*$, $T_h = T_h^*$, $R_h = R_h^*$, $A_m = A_m^*$, $S_m = S_m^*$ and $I_m = I_m^*$. Therefore the largest compact invariant set in $\left\{ S_{h_1}^*, S_{h_2}^*, I_h^*, T_h^*, R_h^*, A_m^*, S_m^*, I_m^* \in \Omega : \frac{dV}{dt} = 0 \right\}$ is the singleton $\{E^*\}$ where E^* is the endemic equilibrium of the model system (3.1). By LaSalle's invariant principle, then it is implies that E^* is globally asymptotically stable in Ω if $A < \mathrm{B}$.This completes the proof.

3.9 Parameter Estimation and Model Validation

In this section, we adopt the idea of Ngailo *et al*., (2014). The work is dedicated to fitting the predicted data of the models in system (3.1) to the Tanzania data obtained from the ministry of health and social welfare in Tanzania and in [http://www.wavuti. com/ 2014 /05/wizara-ya-afya-kitengo-cha.html]. The original data set consists of 5 years observations of the Tanzania dengue fever disease cases from January 2010 to April 2015 in six month interval as shown in Table 3.2.

Table 3. 2: Observed cases of Dengue fever disease between 2010-2015 in Tanzania semi-annually

Period	Number of cases
2010 (Jan-June)	11
2010 (July-December)	29
2011 (Jan-June)	25
2011 (July-December)	20
2012 (Jan-June)	35
2012 (July-December)	15
2013 (Jan-June)	45
2013 (July-December)	32
2014 (Jan-June)	399
2014 (July-December)	624
2015 (Jan-May)	626

Figure 3.3: Shows the trend of dengue fever disease cases in Tanzania

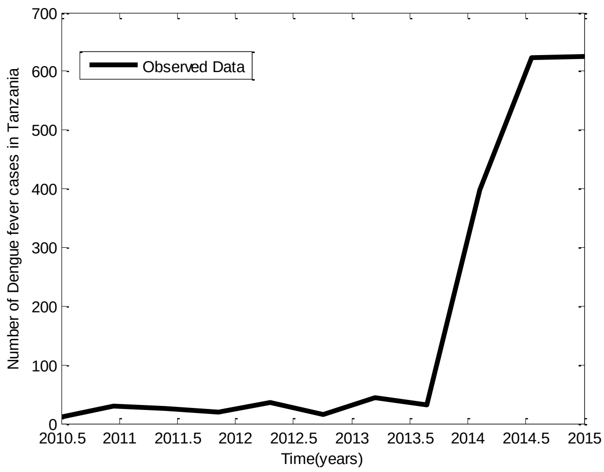


Figure 3. 3: Trend of dengue fever disease in Tanzania

From figure 3.3, it is observed that the number of infected population increases with time.

3.9.1 Parameter Estimation and Evaluation

Maximum likelihood estimation (MLE) is used to determine the best fitting model. The idea is to have a parsimonious model that captures as much variation in the data as possible. Usually the simple graph model captures most of the variability in most stabilized data (Ngailo *et al.*, 2014).

Model system (3.1) is used to the observed data on dengue fever in Tanzania. System (3.1) is fitted to the data for infected individual in Tanzania. Parameter values are obtained from the different literatures and other parameter values are estimated to vary within realistic means and given as shown in the table below.

Table 3. 3 Parameter Values of dengue epidemic

PARAMETER SYMBOL	PARAMETER, VALUES $(yr)^{-1}$	SOURCE
π	0.96	Estimated
B_1	0.09	Estimated
B_2	0.9	Rodrigues *et al.*, (2013)
B_3	0.7	Estimated
μ_m	$\frac{1}{11}$	Dumont, Chiroleu and Domerg, (2008)
k	3	Rodrigues *et al.*, (2013)
β_{hm}	0.375	Rodrigues *et al.*, (2013),Massawe *et al.*, (2015)
β_{mh}	0.39	Estimated
η_A	0.35	Estimated
μ_A	0.25	Estimated
μ_h	$\frac{1}{60\times 365}$	Tanzania bureau of statistics census (2002)
η_h	$\frac{1}{3}$	Rodrigues *et al.*, (2013)
φ	4	Estimated
δ_h	0.9	Estimated
θ_1	0.01	Estimated
θ_2	0.6	Estimated
a	0.005	Estimated

Figure 3.4: Shows dynamic behaviour of predicted and observed data using model system (3.1) and actual data respectively

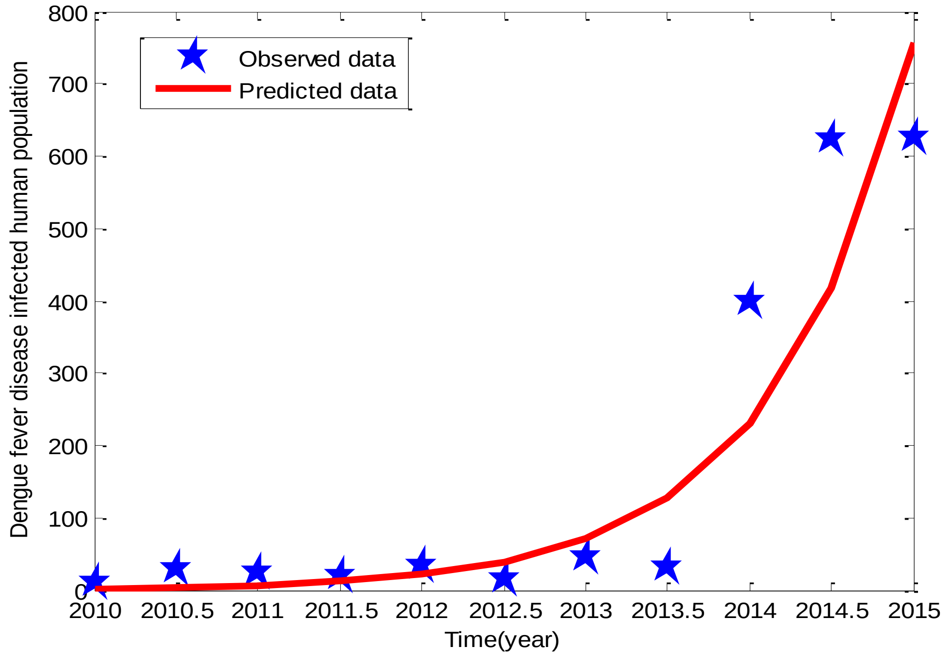


Figure 3. 4: Dynamic behaviour of predicted and observed data using model system (3.1) and actual data respectively

From Figure 3.4: it is observed that, many observed data values are lie within limits few are lie exactly, meaning that the predicted data is similar to the actual data.

Moreover from Figure 3.4, indicate that dengue fever disease in Tanzania is increasing so there is a need to educate the society on how to control the disease. The idea of ML method is to maximize the likelihood function, the likelihood

function is the sum of squares of residual (SSR) defined as $L(\theta)=\sum_{i=1}^{N}\left(y_i-y_i^{est}\right)^2$,

where y_i is a observed data from Tanzania and y_i^{est} is a solution of model equation (3.1) at a parameter value from table 3.3.

Parameters values were estimated by using maximum likelihood estimator (MLE) and obtain the following as shown in Table 3.4

Table 3. 4: Parameter values that give the best fit to the data in the model.

Parameters	**Descriptions**	**Estimated parameter Value**
B_1	Average daily biting for careful susceptible (per day)with $B_2 > B_1$	0.2477
B_2	Average daily biting for careless susceptible	0.8956
B_3	Average daily biting for mosquito	0.6909
β_{mh}	Transmission probability from mosquito to human (per bite)	0.4792
β_{hm}	Transmission probability from human to mosquito (per bite)	0.1431
μ_h	Average lifespan of humans (in days)	$\frac{1}{60\times 365}$
η_h	Mean viremic period (in days)	1.1976
μ_m	Average lifespan of adult mosquitoes (in days)	0.0599
φ	Number of eggs at each deposit per capita (per day)	5.5920
μ_A	Natural mortality of larvae (per day)	0.3964
δ_h	Rate at which dengue fever infected individuals progress for Treatment	0.2502
η_A	Maturation rate from larvae to adult (per day)	0.8076
K	Number of larvae per human	3.8009
π	Fraction of subpopulation recruited into the population.	0.5349
θ_1	Rate at which recovery individuals lose immunity	0.0137
θ_2	Positive change in behaviour of Careless individuals	0.7532
a	Per capita disease induced death rate for humans	0.0105

For the purpose of management and planning of dengue fever disease in Tanzania it is important to project of prevalence of the disease in five years to come of the model. The model predicts that there will be an increase on prevalence of dengue fever disease in Tanzania. The same parameter values are used from Table 3.4.

Figures 3.5 shows the projection of Dengue fever disease infected population in the society of Tanzania up to 2020.

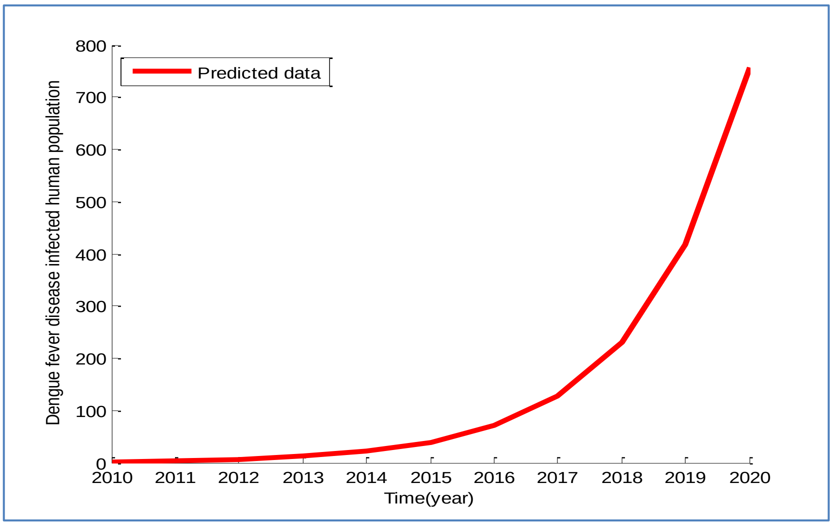


Figure 3. 5: Projection of Dengue fever disease infected population in the society

From Figure 3.5, it is observed that the disease increase slightly at the beginning and increase sharply. This indicates that the application of treatment only is not enough. There is a need to educate society on how to get rid of the disease by using mosquito nets, mosquito repellent or protective clothing, removing vector breeding areas, insecticide application and control maturation rate to adult mosquito.

However statistical test was carried out in order to establish the relationship between forecasted and observed data using SPSS.

Summary for statistics for Tanzania's semiannually observed data is as shown in Table 3.5 (i)-(ii)

Table 3. 5: (i)-(ii): summary for statistics for Tanzania's semiannually observed data.

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
1	0.921[a]	0.848	0.831	97.789

a. Predictors: (Constant), Observed Data

(i)

Coefficients[a]

Model	Unstandardized Coefficients		Standardized Coefficients	T	P-value
	B	Std. Error	Beta		
1 (Constant) Observed Data	6.195 0.870	36.088 0.123	 0.921	0.172 7.075	0.868 0.000

a. Dependent Variable: Predicted Data

(ii)

From Table 3.5 (i) it is observed that coefficient of correlation (R) is +0.921 indicate that there is strong relationship between predicted and observed data as it takes the value between +1 and -1 so +0.921 is close to +1. Coefficient of determination (R Square) is +0.848 showing that there is strong relationship between Predicted and Observed data as it takes the value between 0 and +1 so +0.848 is close to +1. Therefore 85% of the total variability of the predicted data supports the observed data of the model.

From Table 3.5 (ii), $p<0.05$ and then the coefficients in column B which help to predict the model, we have $\text{Predicted data} = 6.195 + 0.870 \times \text{Observed data}$. This implies that the predicted data increases by 0.870 for every unit increase in the observed data.

3.10 Sensitivity Analysis of Model Parameters

In this section we shall determine different factors for disease transmission where this helps to reduce mortality and morbidity due to dengue fever disease.

In order to determine how best human mortality and morbidity due to dengue fever disease is reduced, we calculate the sensitivity indices of the effective reproduction number R_e to each parameter in the model using the approach of Driessche and Watmough, (2002). These indices tell us which parameters have high impact on R_e and should be targeted by adding control strategies (Rodrigues *et al.*, 2013).

Definition 1: The normalised forward sensitivity index of a variable 'p' that depends differentiably on a parameter $'q'$ is defined as:

$$X_q^p = \frac{\partial p}{\partial q} \times \frac{q}{p}$$ (Ratera *et al.*, 2012). (3.49)

Having an explicit formula for R_e in equation (3.49), we derive an analytical expression for the sensitivity of R_e as $X_q^{R_e} = \frac{\partial R_e}{\partial q} \times \frac{q}{R_e}$

where $R_e = \sqrt{\frac{-kB_3\beta_{hm}\beta_{mh}\left(\left(\mu_A+\eta_A\right)\mu_m-\eta_A\varphi\right)\left(B_1\left(1-\pi\right)\left(\mu_h+\theta_2\right)+\theta_2\pi+B_2\pi\mu_h\right)}{\varphi\mu_m^2\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)}}$

Then analytical expression for the sensitivity of R_e with respect to each parameter can be calculated using a set of reasonable parameter values. Parameter values are obtained from the Table 3.4.

The sensitivity indices of R_e with respect to η_A and μ_m are given by $X_{\eta_A}^{R_e} = \frac{\partial R_e}{\partial \eta_A} \times \frac{\eta_A}{R_e}$ $= +0.502671519914268$ and $X_{\mu_m}^{R_e} = \frac{\partial R_e}{\partial \mu_m} \times \frac{\mu_m}{R_e} = -1.0081143036750213$ respectively.

Other indices $X_{B_3}^{R_e}$, $X_{\beta_{hm}}^{R_e}$, $X_{k}^{R_e}$, $X_{\beta_{mh}}^{R_e}$, $X_{\pi}^{R_e}$, $X_{\mu_h}^{R_e}$, $X_{\eta_h}^{R_e}$, $X_{\varphi}^{R_e}$, $X_{B_1}^{R_e}$, $X_{B_2}^{R_e}$, $X_{\theta_2}^{R_e}$, $X_{a}^{R_e}$, $X_{a}^{R_e}$ and $X_{\mu_A}^{R_e}$ are obtained following the same method and tabulated as follows:

Table 3. 6: Sensitivity Indices of Model Parameters to R_e

S/N	Parameter Symbol	Sensitivity Index
1	η_A	+0.502671519914268
2	β_{hm}, B_3, β_{mh} and k	+0.5
3	π	+0.3094913998749647
4	B_1	+0.08860554991815392
5	φ	+0.008114303675021294
6	B_2	+0.0000223354397719466 37
7	θ_2	+0.0000026034820859605 15
8	μ_h	−0.0000215010794076455 5
9	μ_A	−0.002671519914267834
10	a	−0.004345502504120624
11	η_h	−0.4956355998985579
12	μ_m	−1.0081143036750213

The parameters are ordered from most positive sensitive to the least.

3.10.1 Interpretation of Sensitivity Indices

By analysing sensitivity indices of model parameters to R_e, it is observed that the following parameters η_A, β_{hm}, B_3, β_{mh}, k, π, B_1, φ, B_2 and θ_2 when each one increases keeping the other parameters constant they increase the value of R_e implying that they increase the endemicity of the disease as they have positive indices. While parameters such as μ_h, μ_A, a, η_h and μ_m when each one increases while keeping the other parameters constant they decrease the value of R_e implying that they decrease the endemicity of the disease as they have negative indices.

But individually, the most positive sensitive parameter is maturation rate from larvae to adult (per day) η_A, followed by the transmission probability from human to mosquito (per bite) β_{hm}, average daily biting (per day) for mosquito susceptible B_3, transmission probability from mosquito to human (per bite) β_{mh}, number of larvae per human k, Fraction of subpopulation recruited into the population π, average daily biting (per day) for careful human susceptible B_1, number of eggs at each deposit per capita (per day) φ , average daily biting (per day) for careless human susceptible B_2, Positive change in behaviour of Careless individuals θ_2, average lifespan of humans (per day) μ_h, Per capita disease induced death rate for humans a ,natural mortality of larvae (per day) μ_A, mean viremic period (per day) η_h and finally the least positive sensitive parameter is average lifespan of adult mosquitoes (Per day) μ_m .

3.11 Numerical Simulations and Discussion of Results.

In this chapter, we illustrate the analytical results of the study by carrying out numerical simulations of the model system (3.1) using the set of estimated parameter values given in Table 3.4.

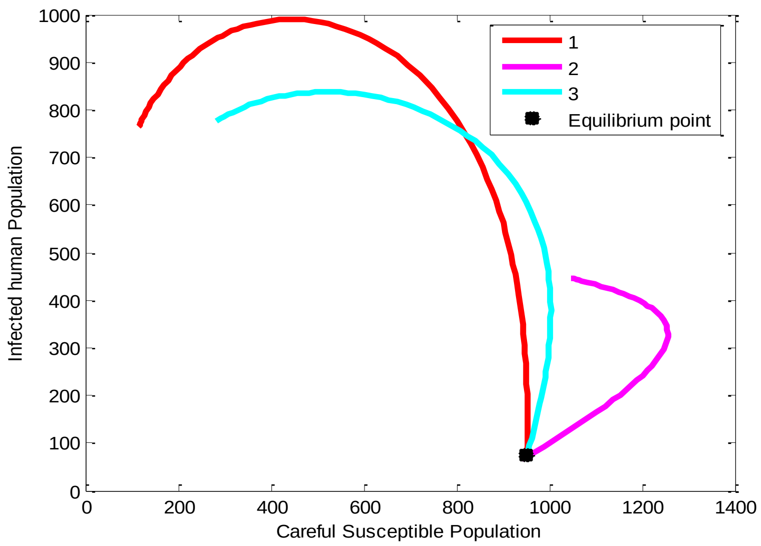
This shows the dynamic behaviour of the endemic equilibrium of the model system (3.1) using the parameter values in Table 3.4 for different initial starting values in three cases as shown below (Ratera *et al*., 2012).

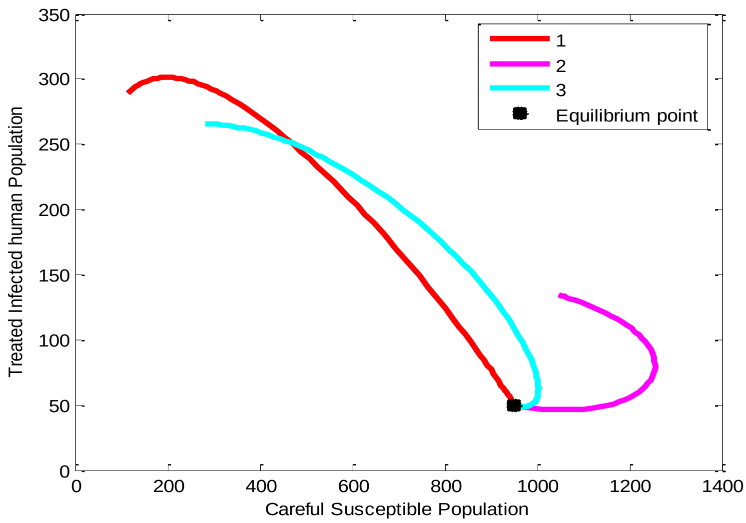
1. $S_{h_1}(0)=951$, $S_{h_2}(0)=950$, $I_h(0)=63$, $T_h(0)=\ 50$, $R_h(0)=49$, $A_m(0)=15000$, $S_m(0)=10000$ and $I_m(0)=5000.$

2. $S_h(0)=940$, $S_{h_2}(0)=945$, $I_h(0)=65$, $T_h(0)=50$, $R_h(0)$=49, $A_m(0)=10000$, $S_m(0)=2000$ and $I_m(0)=1000$

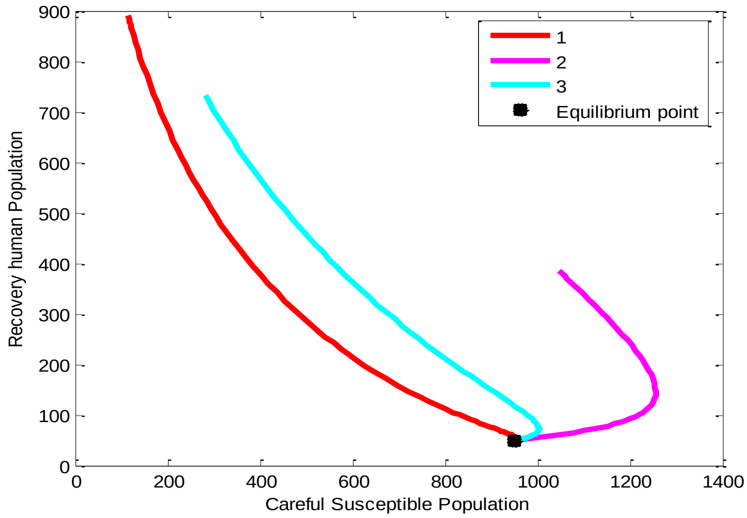
3. $S_h(0)=949$, $S_{h_2}(0)=950$, $I_h(0)=65$, $T_h(0)=50$, $R_h(0)$=49, $A_m(0)=20000$, $S_m(0)=10000$ and $I_m(0)=3000$

Figures 3.6 show the proportion of Dengue fever disease infectives, treated and recovery proportion of Dengue fever disease all plotted against the proportion of susceptible population.

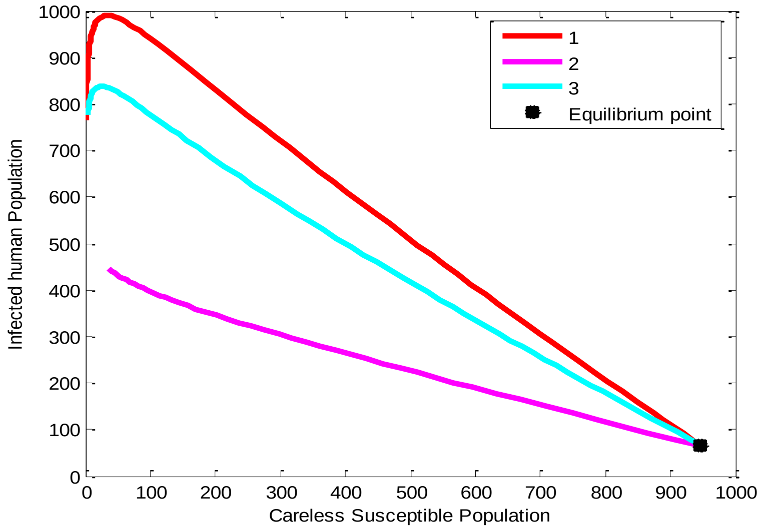




Treated Infected human Population
Careful Susceptible Population
1
2
3
Equilibrium point



Recovery human Population
Careful Susceptible Population
1
2
3
Equilibrium point



Infected human Population
Careless Susceptible Population
1
2
3
Equilibrium point

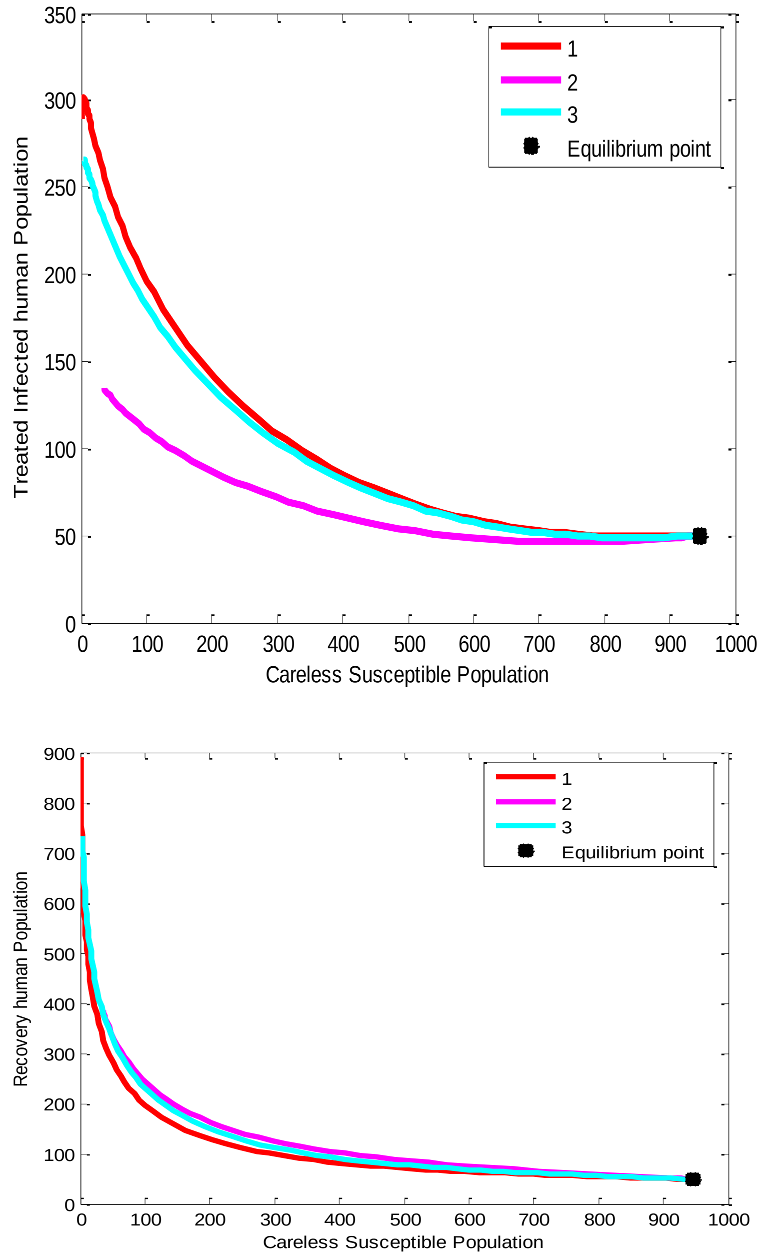


Figure 3. 6: Proportion of Dengue fever disease infectives, treated and recovery proportion of Dengue fever disease

The equilibrium point of the endemic equilibrium E^* was obtained as $S^*_{h_1} = 950$,

$I^*_h = 75$, $T^*_h = 50$ and $R^*_h = 50$ and then $S^*_{h_2} = 946$ $I^*_h = 66$, $T^*_h = 50$ and $R^*_h = 49$

It is observed from Figures 3.6 that for any starting initial value, the solution curves tend to the equilibrium E^*. Therefore we conclude that the model system (3.1) is globally stable about this endemic equilibrium point E^* for the parameters displayed in Table 3.4

Figures 3.7: Shows the distribution of population with time in all classes of human and mosquito.

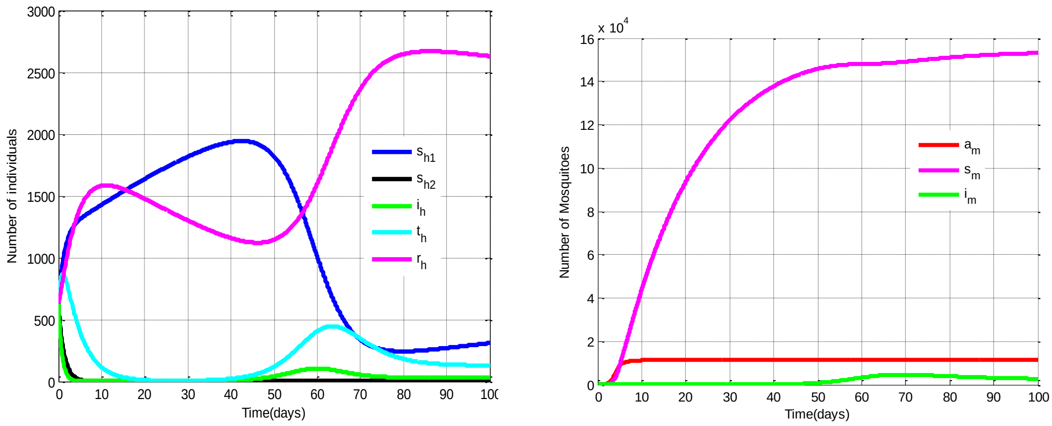


Figure 3. 7: Distribution of population with time in all classes of human and mosquito.

From Figures 3.7, it observed that human infection reaches a peak between the 45th and the 75th day. The infection of the mosquitoes delayed. The total number of infected humans obtained from System (3.1) is lower than observations in Tanzania. The difference is due to the absence of the data in the whole country of Tanzania (Rodrigues *et al.*, 2012).

Figure 3.8 (a)-(d) shows the variation of susceptible and infected mosquito population for different values of maturation rate from larvae to adult (per day) η_A.

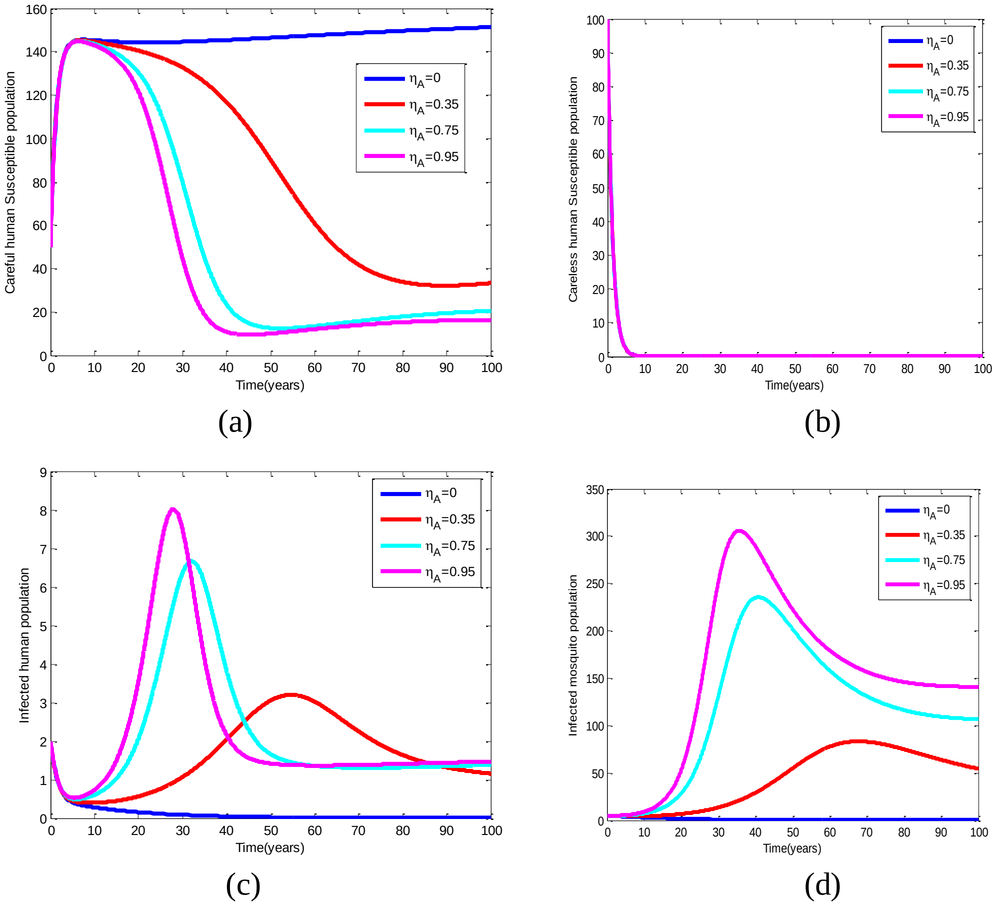


Figure 3. 8: (a)-(d): Variation of susceptible, infected human and mosquito population for different values of maturation rate from larvae to adult (per day).

From figure 3.8 (a)-(d), we vary the maturation rate from larvae to adult (per day) η_A , and it is observed that when maturation rate from larvae to adult (per day) increases, careful and careless susceptible decrease while infected human and mosquito population increased.

3.12 Summary

A compartmental model for Dengue fever disease was presented, based on two populations, humans (with temporary immunity, careful and careless susceptible) and mosquitoes with treatment. Sensitivity analysis revealed parameters which have high impact on R_e or dengue fever disease transmission. These parameters should be targeted by adding control strategies, the parameters are as follows ; the most positive sensitive parameter is maturation rate from larvae to adult (per day) η_A, followed by the transmission probability from human to mosquito (per bite) β_{hm}, average daily biting (per day) for mosquito susceptible B_3, transmission probability from mosquito to human (per bite) β_{mh}, number of larvae per human k, Fraction of subpopulation recruited into the population π, number of eggs at each deposit per capita (per day) φ, Positive change in behaviour of careless individuals θ_2,average daily biting (per day) for careless human susceptible B_2. From this results in order to reduce mortality and morbidity due to dengue fever disease we need to control the above parameters using different strategy like campaign aimed in educating careless individuals as a mean of minimizing or eliminating mosquito-human contact, control effort aimed at reducing mosquito-human contact, the control effort for removing vector breeding places, insecticide application and the control effort aimed at reducing the maturation rate from larvae to adult in order to reduce the number of infected individual, however there is cost associated with this control, implying the need of minimizing the costs as financial resources are always scarce. In doing this, Optimal Control (OC) approach using Pontryagin's Maximum principle is preferred in order to find the best strategy to fight the disease and minimize the cost.

In validation of the model, it is observed that from Figure 3.4 many observed data values lie within the limits. Few lie exactly on the curve, implying that the predicted data agree with the actual data. Moreover from Figure 3.5, the projection indicates that dengue fever disease in Tanzania is increasing so there is a need to educate the society on how to control the disease by removing vector breading area, insecticide application, control maturation rate from larvae to adult mosquito, and reduce vector-human contact by using mosquito nets, mosquito repellent or protective clothing.

CHAPTER FOUR

MODEL OF DENGUE FEVER DISEASE WITH CONTROLS

4.1 Extended Model with Controls

In this chapter, an optimal control problem is formulated by incorporating control strategies into the model of dengue fever with treatment, temporary immunity, careful and careless human susceptible. The following controls are incorporated into the model: campaign aimed at educating careless susceptible individuals u_1, reducing mosquito-human contact u_2, removing vector breeding places u_3, insecticide application u_4 and reducing the maturation rate from larvae to adult u_5. However there is cost associated with this controls, so there is a need of minimizing the costs as financial resources are always scarce. In doing this, Optimal Control (OC) approach using Pontryagin's Maximum principle is preferred in order to find the best strategy to fight the disease and minimize the cost in Tanzania.

4.2 Formulation of the Model

In this section, a deterministic model is developed that describes the dynamics of Dengue fever of population size N (Rodrigue *et al.,* 2013). Two types of population are considered: humans and mosquito. The humans are divided into five mutually-exclusive compartments indexed by h are given by: $S_{h_1}(t)$- careful human susceptible (Individual who are aware of the disease and use protective measure), $S_{h_2}(t)$- careless human susceptible (Individual who are not aware of the disease and are not using protective measure), where the biting rate and infected with the disease for careless susceptible is higher than that of careful susceptible individual. Careless susceptible individual may change in behaviour to be careful at a rate of θ_2, $I_h(t)$- individuals capable of transmitting dengue fever disease to others; $T_h(t)$- individual who are treated and $R_h(t)$- individuals who have acquired immunity at time *t*. The total number of human is constant, which means that $N(t) = S_{h_1}(t) + S_{h_2}(t) + I_h(t) + T_h(t) + R_h(t)$.

Hence we formulate the $S_{h_{1,2}}ITRS_{h_1}$ model to describe the passage of individual from careful or careless susceptible class $S_{h_{1,2}}(t)$, to the infected class $I_h(t)$, to the treatment class $T_h(t)$, to the recovery class $R_h(t)$, and then to the careful susceptible class S_{h_1}, indicating that individual lose immunity on recovery from the infection class. Careful and careless susceptible individual are obtained in the population at a constant rate of $1-\pi$ and π respectively, it assumed that immigration and emigration are not considered and then the population is homogeneous, which means that every individual of a compartment is homogeneously mixed with the other individuals.

Individual acquire dengue fever infection after infected with dengue virus from mosquito biting rate B_1 and B_2 for careful and careless susceptible with the force of infection $\lambda_1 = (1-u_2)B_1\beta_{mh}\frac{I_m}{N_h}$ and $\lambda_2 = (1-u_2)B_2\beta_{mh}\frac{I_m}{N_h}$ respectively, β_{mh} is infection from mosquito to human. It is assumed that $B_2 > B_1$, there is no natural protection for human and mosquito.

However infected individual can die with the disease at the rate of a ,or can move to the other class which is treatment at the rate of η_h .

Furthermore dengue fever infected individual progress for treatment at the rate of δ_h , where the treated class move to the recovery class at a time t and then lose immunity at a rate of θ_1 . Human classes are assumed to die naturally at a rate of μ_h . Similarly, the model has also three compartments for the mosquito (mosquitoes) indexed by m are given by: $A_m(t)$, which represents the aquatic phase of the mosquito (including egg, pupae and larvae) and the adult phase of the mosquito, with $S_m(t)$ and $I_m(t)$, susceptible and infected, respectively. It is also assumed that $N_m(t) = S_m(t) + I_m(t)$. Then also we formulate the $A_m S_m \mathrm{I}_m$ model to describe the passage from the aquatic phase of the mosquito $A_m(t)$, to the the adult phase of susceptible mosquito $S_m(t)$ and to the the adult phase of infected mosquito $I_m(t)$.

The eggs are obtained from either susceptible or infected mosquito at the rate of φ , for which aquatic phase will mature to adult at the rate of η_A , susceptible mosquito will be infected with dengue virus after biting infected human at the rate of B_3, with the force of infection $\lambda_3 = (1-u_2) B_3 \beta_{mh} \dfrac{I_m}{N_h}$, where, β_{hm} is transmission probability from human to mosquito. It also assumed that each vector has an equal probability to bite any host and there is no resistant phase, due to its short lifetime. Furthermore aquatic phase $A_m(t)$ and adult phase of mosquito can die naturally at the rate of μ_A and μ_m respectively.

Then we include five controls that is u_1 - the control aimed at changing behaviour of careless human susceptible (campaign aimed in educating careless individual), u_2 - control aimed at reducing mosquito-human contact, u_3 -control aimed for removing vector breeding places, u_4 - insecticide application and u_5 - control aimed at reducing the maturation rate from larvae to adult. These control functions u_1 , u_2 , u_3 , u_4 and u_5 are bounded and Lebesgue integrable.

Considering the above assumptions and considerations, we then have the following schematic model flow diagram for dengue fever disease with controls:

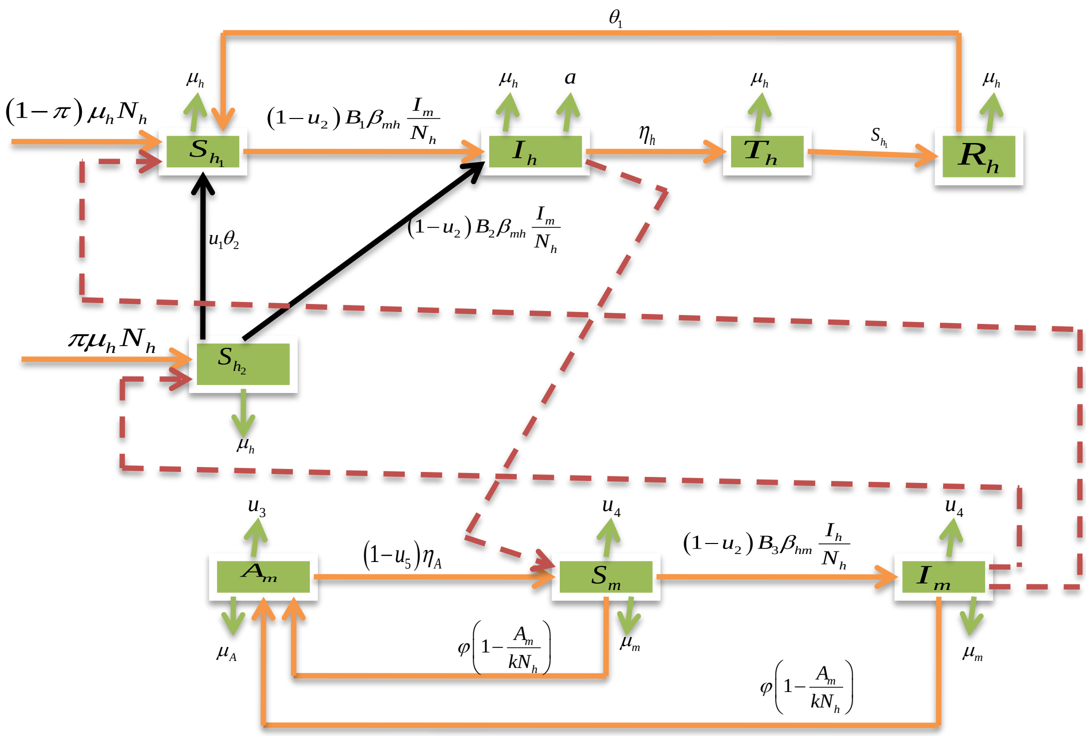


Figure 4. 1: Model Flow diagram for dengue fever disease with control

From the above flow diagram, the model will be governed by the following equations:

$$
\begin{aligned}
\frac{dS_{h_1}}{dt} &= (1-\pi)\mu_h N_h - (1-u_2)B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + u_1\theta_2 S_{h_2} \\
\frac{dS_{h_2}}{dt} &= \pi\mu_h N_h - (1-u_2)B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - (\mu_h + u_1\theta_2)S_{h_2} \\
\frac{dI_h}{dt} &= \left((1-u_2)B_1 S_{h_1} + (1-u_2)B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h \\
\frac{dT_h}{dt} &= \eta_h I_h - (\mu_h + \delta_h)T_h \\
\frac{dR_h}{dt} &= \delta_h T_h - (\mu_h + \theta_1)R_h \\
\frac{dA_m}{dt} &= \varphi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - \left(\mu_A + (1-u_5)\eta_A + u_3\right)A_m \\
\frac{dS_m}{dt} &= (1-u_5)\eta_A A_m - \left((1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}\right)S_m - (\mu_m + u_4)S_m \\
\frac{dI_m}{dt} &= (1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}S_m - (\mu_m + u_4)I_m
\end{aligned}
\tag{4.1}
$$

4.3 Model Analysis

The model system of equations (4.1) will be analysed qualitatively to get insight into its dynamical features which will give a better understanding of the Modelling Infectiology and Optimal Control of Dengue Epidemics in the society. Threshold which governs elimination or persistence of Dengue fever will be determined and studied. We begin by finding the invariant region and show that all solutions of system (4.1) are positive for all the time.

4.3.1 Positive Invariant Region of the Model

Since the model system (4.1) is Dengue fever disease model dealing with human population, we assume that all state variables and parameters of the model are positive for all $t \geq 0$. The model (4.1) will be analysed in suitable feasible region where all state variables are positive. This region is contained in $\Omega \in R_+^8$.

Let $\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right) \in R_+^8$ be any solution of the system (4.1) given by $S_{h_1} + S_{h_2} + I_h + T_h + R_h \leq N_h$, $A_m \leq kN_h$ & $S_m + I_m \leq mN_h$ with non-negative initial conditions, (Rodrigues *et al.* 2013). Then the solutions $\{S_{h_1}(t),\ S_{h_2}(t),\ I_h(t),\ T_h(t)$, $R_h(t),\ A_m(t),\ S_m(t),\ I_m(t)\}$ of the system (4.1) are positive for all the time.

Proof:

First, with regards of the population of human whose total is denoted by N_h , we have $N_h = S_{h_1} + S_{h_2} + I_h + T_h + R_h$ and the system of differential equations is given by $\frac{dN_h}{dt} = \frac{dS_{h_1}}{dt} + \frac{dS_{h_2}}{dt} + \frac{dI_h}{dt} + \frac{dT_h}{dt} + \frac{dR_h}{dt}$.

Using system of equations (4.1), we get

$$\begin{aligned}\frac{dN_h}{dt} &= (1-\pi)\mu_h N_h - (1-u_2)B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + u_1\theta_2 S_{h_2} \\ &\quad + \pi\mu_h N_h - (1-u_2)B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - (\mu_h + u_1\theta_2)S_{h_2} \\ &\quad + \left((1-u_2)B_1 S_{h_1} + (1-u_2)B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h \\ &\quad - (\mu_h + \delta_h)T_h + \delta_h T_h - (\mu_h + \theta_1)R_h\end{aligned}$$

This gives

$$\frac{dN_h}{dt} \geq \mu_h N_h - \mu_h\left(S_{h_1} + S_{h_2} + I_h + T_h + R_h\right).$$

Then

$$\frac{dN_h}{dt} \geq \mu_h N_h - \mu_h N_h$$

or

$$\frac{dN_h}{dt} \geq 0$$

$$dN_h \geq dt$$

Integrating both sides, gives

$$N_h \geq c$$

Since

$$N_h \geq S_{h_1} + S_{h_2} + I_h + T_h + R_h$$

it follows that

$$N_h \geq S_{h_1} + S_{h_2} + I_h + T_h + R_h \geq c.$$

Hence

$$S_{h_1} + S_{h_2} + I_h + T_h + R_h \leq N_h.$$

Secondly, with regards of the population of aquatic phase denoted by A_m, we have

$$\frac{dA_m}{dt} = \varphi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - \left(\mu_A + (1-u_5)\eta_A + u_3\right)A_m$$

or

$$\frac{dA_m}{dt} \geq -\left(\mu_A + (1-u_5)\eta_A + u_3\right)A_m$$

This is the first order inequality which can be solved using separation of variables
Separate the variable yields

$$\frac{dA_m}{A_m} \geq -\left(\mu_A + (1-u_5)\eta_A + u_3\right)dt$$

Integrating both sides one gets

$$\ln A_m \geq -\left(\mu_A + (1-u_5)\eta_A + u_3\right)t + \ln C$$

This is equivalent to

$$A_m(t) \geq Ce^{-\left(\mu_A + (1-u_5)\eta_A + u_3\right)t}.$$

As $t \to 0$, $A_m(0) \geq C.$

Then $A_m(t) \geq A_m(0)e^{-\left(\mu_A + (1-u_5)\eta_A + u_3\right)t}$

As $t \to \infty$, $A_m(t) \geq 0$

Then

$$0 \leq A_m(t) \leq kN_h$$

Finally, with regards of the total population of mosquito, we have

$$N_m = S_m + \mathrm{I}_m$$

and the system of differential equations, is given by

$$\frac{dN_m}{dt} = \frac{dS_m}{dt} + \frac{dI_m}{dt}.$$

But from the system (4.1) we have

$$\frac{dS_m}{dt} = (1-u_5)\eta_A A_m - \left((1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}\right)S_m - (\mu_m + u_4)S_m$$

and

$$\frac{dI_m}{dt} = (1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}S_m - (\mu_m + u_4)I_m$$

Then, $\frac{dN_m}{dt} = (1-u_5)\eta_A A_m - \left((1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}\right)S_m - (\mu_m + u_4)S_m$

$$+(1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}S_m - (\mu_m + u_4)I_m \qquad \text{or}$$

$$\frac{dN_m}{dt} \geq -(\mu_m + u_4)(S_m + I_m)$$

It follows that

$$\frac{dN_m}{dt} \geq -(\mu_m + u_4)N_m$$

This is the first order inequality which can be solved using separation of variable.

Then

$$\frac{dN_m}{N_m} \geq -(\mu_m + u_4)dt.$$

Integrating both sides gives

$$\ln N_m \geq -(\mu_m + u_4)t + \ln C.$$

This is equivalent to

$$N_m(t) \geq Ce^{-(\mu_m + u_4)t}.$$

As $t \to 0 \quad N_m(0) \geq C$.

Then, $N_m(t) \geq N_m(0)e^{-(\mu_m+u_4)t}$

AS $t \to \infty \quad N_m(t) \geq 0.$

Then,

$0 \leq N_m \leq S_m + I_m \leq mN_h$

Therefore $0 \leq S_m + I_m \leq mN_h$

Hence the host population size $S_{h_1} + S_{h_2} + I_h + T_h + R_h \leq N_h$. For Aquatic phase (that includes the egg, larva and pupae stages), the total population size $A_m \leq kN_h$ as $t \to \infty$.

Finally, for vector the total population size, $S_m + I_m \leq mN_h$ as $t \to \infty$

Therefore the feasible set for the model system (4.1) is given by

$$\Omega = \left\{ \begin{array}{l} (S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m) \in R_+^8 : S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m \geq 0, \\ S_{h_1} + S_{h_2} + I_h + T_h + R_h \leq N_h, A_m \leq kN_h, S_m + I_m \leq mN_h \end{array} \right\}$$

Hence it is verified that Ω is a positively invariant set with respect to (4.1).

4.3.2 Positivity of the solutions

Since model system (4.1) dealing with human population, it is assumed that all state variables and parameters of the model are positive for all $t \geq 0$. For the system (4.1) to be epidemiologically meaningful, it will be proved that all solutions with non-negative initial data will remain non-negative that is $S_{h_1}(0)$, $S_{h_2}(0)$, $I_h(0)$, $T_h(0)$, $R_h(0)$, $A_m(0)$, $S_m(0)$ & $I_m(0)$ are non-negative. We prove by the following Lemma:

Lemma 4.1

Let

$\left\{S_{h_1}(0)\geq 0, S_{h_2}(0)\geq 0, I_h(0)\geq 0, T_h(0)\geq 0, R_h(0)\geq 0, A_m(0)\geq 0, S_m(0)\geq 0 \text{ and } I_m(0)\geq 0\right\}\in \Omega$

Then the solution set $\left\{S_{h_1}(t), S_{h_2}(t), I_h(t), T_h(t), R_h(t), A_m(t), S_m(t), I_m(t)\right\}$ of the model system (4.1) is positive, for all $t \geq 0$. (Ratera *et al.*, 2012)

Proof:

To prove the Lemma, we shall use all the equations of the system (4.1).

From the 1^{st} equation of the system (4.1) we have

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - (1-u_2)B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + u_1\theta_2 S_{h_2}$$

or

$$\frac{dS_{h_1}}{dt} \leq (1-\pi)\mu_h N_h - \mu_h S_{h_1}$$

Consequently

$$\frac{dS_{h_1}}{dt} + \mu_h S_{h_1} \leq (1-\pi)\mu_h N_h.$$

This is a first order inequality which can be solved using an integrating factor I.F: $e^{\mu_h \int dt} = e^{\mu_h t}$ Multiply the inequality by the I.F on both sides, we get

$$e^{\mu_h t}\frac{dS_{h_1}}{dt} + e^{\mu_h t}\mu_h S_{h_1} \leq e^{\mu_h t}(1-\pi)\mu_h N_h.$$

This is equivalent to

$$d\left(e^{\mu_h t}S_{h_1}(t)\right) \leq (1-\pi)\mu_h N_h e^{\mu_h t}dt.$$

Integrate both sides yields

$$e^{\mu_h t}S_{h_1}(t) \leq (1-\pi)N_h e^{\mu_h t} + C.$$

This gives

$$S_{h_1}(t) \leq (1-\pi)N_h + Ce^{-\mu_h t}.$$

As $t \to 0$, it follows that $S_{h_1}(0) \leq (1-\pi)N_h + C$ or

$$S_{h_1}(0)-(1-\pi)N_h \le C.$$

Consequently

$$S_{h_1}(t)\le(1-\pi)N_h+\left(S_h(0)-(1-\pi)N_h\right)e^{-\mu_h t}$$

As $t\to\infty$ then $S_{h_1}(t)\le(1-\pi)N_h$. Therefore $0\le s_{h_1}(t)\le(1-\pi)N_h$.

From the second equation of the system (4.1), we have

$$\frac{dS_{h_2}}{dt}=\pi\mu_h N_h-(1-u_2)B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2}-\left(\mu_h+u_1\theta_2\right)S_{h_2} \text{ or}$$

$$\frac{dS_{h_2}}{dt}\le\pi\mu_h N_h-\left(\mu_h+u_1\theta_2\right)S_{h_2}$$

Consequently

$$\frac{dS_{h_2}}{dt}+\left(\mu_h+u_1\theta_2\right)S_{h_2}\le\pi\mu_h N_h.$$

This is a first order inequality which can be solved using an integrating factor I.F

$e^{(\mu_h+u_1\theta_2)\int dt}=e^{(\mu_h+u_1\theta_2)t}$ Multiply the inequality by the I.F on both sides to get

$$e^{(\mu_h+u_1\theta_2)t}\frac{dS_{h_2}}{dt}+e^{(\mu_h+u_1\theta_2)t}\mu_h S_{h_1}\le e^{(\mu_h+u_1\theta_2)t}\pi\mu_h N_h.$$

This is equivalent to

$$d\left(e^{(\mu_h+u_1\theta_2)t}S_{h_2}(t)\right)\le\pi\mu_h N_h e^{(\mu_h+u_1\theta_2)t}dt.$$

Integrating both sides yields

$$e^{(\mu_h+u_1\theta_2)t}S_{h_2}(t)\le\pi N_h e^{(\mu_h+u_1\theta_2)t}+C.$$

This gives $S_{h_2}(t)\le\pi N_h+Ce^{-(\mu_h+u_1\theta_2)t}$.

As $t\to 0$, it follows that $S_{h_2}(0)\le\pi N_h+C$ or $S_{h_2}(0)-\pi N_h\le C.$

Consequently

$$S_{h_2}(t) \le \pi N_h + \left(S_{h_2}(0) - \pi N_h\right) e^{-(\mu_h + u_1\theta_2)t}$$

As $t \to \infty$ then $S_{h_2}(t) \le \pi N_h$. Therefore $0 \le s_{h_2}(t) \le \pi N_h$.

From the third equation of the system (4.1), we have

$$\frac{dI_h}{dt} = \left((1-u_2)B_1 S_{h_1} + (1-u_2)B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h$$

or

$$\frac{dI_h}{dt} \ge -(\mu_h + \eta_h + a)I_h$$

This is the first order inequality which can be solved using separation of variable. Then

$$\frac{dI_h}{I_h} \ge -(\mu_h + \eta_h + a)dt$$

Integrate both sides gives

$$\ln I_h \ge -(\mu_h + \eta_h + a)t + \ln C,$$

which is equivalent to

$$I_h(t) \ge Ce^{-(\mu_h + \eta_h + a)t}$$

As $t \to 0$ it follows that $I_h(0) \ge C$

Consequently $I_h(t) \ge I_h(0)e^{-(\mu_h + \eta_h + a)t}$

As $t \to \infty$, it follows that $I_h(t) \ge 0$.

Therefore $I_h(t) \ge 0 \quad \forall t \ge 0$

Similarly, it is shown that the remaining five equations of system (4.1) are all positive.

Therefore it is true that $S_{h_1}(t)\ge 0$, $S_{h_2}(t)\ge 0$, $I_h(t)\ge 0$, $T_h(t)\ge 0$, $R_h(t)\ge 0$, $A_m(t)\ge 0$, $S_m(t)\ge 0$ and $I_m(t)\ge 0$, $\forall t\ge 0$

4.4 Steady State Solutions

In this section the model system (4.1) is qualitatively analysed by determining the model equilibria, carrying out their corresponding stability analysis and interpreting the results. Let $E=\left(S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*,I_m^*\right)$ be the equilibrium point of the system (4.1). Then, setting the right hand side of system (4.1) to zero, we obtain

$$(1-\pi)\mu_h N_h-\lambda_1^* S_{h_1}^*-\mu_h S_{h_1}^*+\theta_1 R_h^*+u_1\theta_2 S_{h_2}^*=0 \tag{4.2}$$

$$\pi\mu_h N_h-\lambda_2^* S_{h_2}^*-\left(\mu_h+u_1\theta_2\right)S_{h_2}^*=0 \tag{4.3}$$

$$\left(\lambda_1^* S_{h_1}^*+\lambda_2^* S_{h_2}^*\right)-\left(\mu_h+\eta_h+a\right)I_h^*=0 \tag{4.4}$$

$$\eta_h I_h^*-\left(\mu_h+\delta_h\right)T_h^*=0 \tag{4.5}$$

$$\delta_h T_h^*-\left(\mu_h+\theta_1\right)R_h^*=0 \tag{4.6}$$

$$\varphi\left(1-\frac{A_m^*}{kN_h}\right)\left(S_m^*+I_m^*\right)-\left(\mu_A+u_3+\left(1-u_5\right)\eta_A\right)A_m^*=0 \tag{4.7}$$

$$\left(1-u_5\right)\eta_A A_m^*-\left(\lambda_3^*+\mu_m+u_4\right)S_m^*=0 \tag{4.8}$$

$$\lambda_3^* S_m^*-\left(\mu_m+u_4\right)I_m^*=0 \tag{4.9}$$

Forces of infections are

$$\lambda_1^*=\left(1-u_2\right)B_1\beta_{mh}\frac{I_m^*}{N_h} \tag{4.10}$$

$$\lambda_2^* = (1-u_2)B_2\beta_{mh}\frac{I_m^*}{N_h} \tag{4.11}$$

$$\lambda_3^* = (1-u_2)B_3\beta_{hm}\frac{I_h^*}{N_h} \tag{4.12}$$

We compute all state variables of dengue fever disease model in terms of force of infection λ^*

From equation (4.2) we have

$$S_{h_1}^* = \frac{(1-\pi)\mu_h N_h + \theta_1 R_h^* + u_1\theta_2 S_{h_2}^*}{(\lambda_1^* + \mu_h)} \tag{4.13}$$

The we substitute (4.6) into (4.13) to get

$$S_{h_1}^* = \frac{(1-\pi)\mu_h N_h + \theta_1\dfrac{\delta_h\eta_h(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*)}{(\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)} + u_1\theta_2 S_{h_2}^*}{(\lambda_1^* + \mu_h)}$$

or

$$S_{h_1}^* = \frac{(1-\pi)\mu_h N_h(\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a) + \theta_1\delta_h\eta_h(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*) + (\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)u_1\theta_2 S_{h_2}^*}{(\lambda_1^* + \mu_h)(\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)}$$

where, $R_h^* = \dfrac{\delta_h T_h^*}{(\mu_h+\theta_1)}$ from (4.6),

$T_h^* = \dfrac{\eta_h I_h^*}{(\mu_h+\delta_h)}$ from (4.5),

$I_h^* = \dfrac{(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*)}{(\mu_h+\eta_h+a)}$ from (4.4)

Then $R_h^* = \dfrac{\delta_h\eta_h I_h^*}{(\mu_h+\theta_1)(\mu_h+\delta_h)}$ or $R_h^* = \dfrac{\delta_h\eta_h(\lambda_1^* S_{h_1}^* + \lambda_2^* S_{h_2}^*)}{(\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)}$

It follows that

$$\left(\left(\lambda_1^*+\mu_h\right)\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)-\theta_1\delta_h\eta_h\lambda_1^*\right)S_{h_1}^* =$$

$$\left(1-\pi\right)\mu_h N_h\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)+\theta_1\delta_h\eta_h\lambda_2^* S_{h_2}^*$$
$$+\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)u_1\theta_2 S_{h_2}^*$$

or

$$S_{h_1}^* = \frac{\left(1-\pi\right)\mu_h N_h\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)+\theta_1\delta_h\eta_h\lambda_2^* S_{h_2}^* +\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)u_1\theta_2 S_{h_2}^*}{\left(\left(\lambda_1^*+\mu_h\right)\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)-\theta_1\delta_h\eta_h\lambda_1^*\right)}$$

Then we substitute (4.3) to get

$$S_{h_1}^* = \frac{\left(1-\pi\right)\mu_h N_h\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)\left(\lambda_2^*+\left(\mu_h+u_1\theta_2\right)\right)+\theta_1\delta_h\eta_h\lambda_2^*\pi\mu_h N_h +\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)u_1\theta_2\pi\mu_h N_h}{\left(\lambda_2^*+\left(\mu_h+u_1\theta_2\right)\right)\left(\left(\lambda_1^*+\mu_h\right)\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)-\theta_1\delta_h\eta_h\lambda_1^*\right)}$$

Let $\quad f_1=\left(1-\pi\right)\mu_h N_h\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)$

$$f_3=\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right)u_1\theta_2\pi\mu_h N_h$$

$$f_2=\left(\mu_h+u_1\theta_2\right)\ f_4=\left(\mu_h+\theta_1\right)\left(\mu_h+\delta_h\right)\left(\mu_h+\eta_h+a\right).\ f_5=\theta_1\delta_h\eta_h\pi\mu_h N_h$$

Consequently

$$S_{h_1}^* = \frac{f_1\left(\lambda_2^*+f_2\right)+f_5\lambda_2^*+f_3}{\left(\lambda_2^*+f_2\right)\left(\left(\lambda_1^*+\mu_h\right)f_4-\theta_1\delta_h\eta_h\lambda_1^*\right)} \tag{4.14}$$

From (4.3) we have

$$S_{h_2}^* = \frac{\pi\mu_h N_h}{\left(\lambda_2^*+f_2\right)} \tag{4.15}$$

From equation (4.5) we have

$$I_h^* = \frac{\left(\lambda_1^* S_{h_1}^*+\lambda_2^* S_{h_2}^*\right)}{\left(\mu_h+\eta_h+a\right)}$$

Then substitute (4.14) and (4.15) to obtain

$$I_h^* = \frac{\left(\lambda_1^* \frac{f_1\left(\lambda_2^* + f_2\right) + f_5\lambda_2^* + f_3}{\left(\lambda_2^* + f_2\right)\left(\left(\lambda_1^* + \mu_h\right) f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)} + \lambda_2^* \frac{\pi\mu_h N_h}{\left(\lambda_2^* + f_2\right)} \right)}{\left(\mu_h + \eta_h + a\right)}$$

or

$$I_h^* = \frac{\lambda_1^*\left(f_1\left(\lambda_2^* + f_2\right) + f_5\lambda_2^* + f_3\right) + \left(\left(\lambda_1^* + \mu_h\right) f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h}{\left(\lambda_2^* + f_2\right)\left(\left(\lambda_1^* + \mu_h\right) f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h + \eta_h + a\right)} \tag{4.16}$$

substitute (4.16) in (4.5) to get

$$T_h^* = \frac{\eta_h I_h^*}{\left(\mu_h + \delta_h\right)}$$

or

$$T_h^* = \frac{\lambda_1^*\eta_h\left(f_1\left(\lambda_2^* + f_2\right) + f_5\lambda_2^* + f_3\right) + \left(\left(\lambda_1^* + \mu_h\right) f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h}{\left(\mu_h + \delta_h\right)\left(\lambda_2^* + f_2\right)\left(\left(\lambda_1^* + \mu_h\right) f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h + \eta_h + a\right)} \tag{4.17}$$

Substitute (4.17) in (4.6) to get

$$R_h^* = \frac{\delta_h T_h^*}{\left(\mu_h + \theta_1\right)}$$

or

$$R_h^* = \frac{\lambda_1^*\delta_h\eta_h\left(f_1\left(\lambda_2^* + f_2\right) + f_5\lambda_2^* + f_3\right) + \left(\left(\lambda_1^* + \mu_h\right) f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h}{\left(\mu_h + \theta_1\right)\left(\mu_h + \delta_h\right)\left(\lambda_2^* + f_2\right)\left(\left(\lambda_1^* + \mu_h\right) f_4 - \theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h + \eta_h + a\right)} \tag{4.18}$$

Then from (4.8) and (4.9) we have

$$S_m^* = \frac{(1 - u_5)\eta_A A_m^*}{\left(\lambda_3^* + \mu_m + u_4\right)} \tag{4.19}$$

$I_m^* = \dfrac{\lambda_3^* S_m^*}{\left(\mu_m + u_4\right)}$ which is equivalent to

$$I_m^* = \frac{\lambda_3^*(1-u_5)\eta_A A_m^*}{(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)} \tag{4.20}$$

Substitute (4.19) and (4.20) into (4.7) to obtain

$$\varphi\left(1-\frac{A_m^*}{kN_h}\right)\left(\frac{(1-u_5)\eta_A A_m^*}{(\lambda_3^*+\mu_m+u_4)}+\frac{\lambda_3^*(1-u_5)\eta_A A_m^*}{(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)}\right)-\left(\mu_A+u_3+(1-u_5)\eta_A\right)A_m^*=0$$

or

$$1-\frac{A_m^*}{kN_h}=\frac{\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)}{\varphi\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)}$$

Consequently

$$A_m^*=\frac{kN_h\begin{pmatrix}\varphi\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)\\ -\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)\end{pmatrix}}{\varphi\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)} \tag{4.21}$$

Then we substitute (4.21) into (4.19) and (4.20) to obtain

$$S_m^*=\frac{kN_h(1-u_5)\eta_A\begin{pmatrix}\varphi\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)\\ -\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)\end{pmatrix}}{\varphi(\lambda_3^*+\mu_m+u_4)\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)} \tag{4.22}$$

$$I_m^*=\frac{\lambda_3^*(1-u_5)\eta_A kN_h\begin{pmatrix}\varphi\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)\\ -\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)\end{pmatrix}}{\varphi(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)} \tag{4.23}$$

Furthermore we substitute (4.16) and (4.23) into (4.10) to (4.12) to yield

Force of infections

$$\lambda_1^*=\frac{(1-u_2)B_1\beta_{mh}\lambda_3^*(1-u_5)\eta_A kN_h\begin{pmatrix}\varphi\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)\\ -\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)\end{pmatrix}}{\varphi N_h(\mu_m+u_4)(\lambda_3^*+\mu_m+u_4)\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)} \tag{4.24}$$

$$\lambda_2^* = \frac{(1-u_2)B_2\beta_2\lambda_3^*(1-u_5)\eta_A k N_h \begin{pmatrix} \varphi\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right) \\ -\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)\left(\lambda_3^*+\mu_m+u_4\right) \end{pmatrix}}{\varphi N_h(\mu_m+u_4)\left(\lambda_3^*+\mu_m+u_4\right)\left((\mu_m+u_4)(1-u_5)\eta_A+\lambda_3^*(1-u_5)\eta_A\right)} \qquad (4.25)$$

$$\lambda_3^* = \frac{\begin{matrix}(1-u_2)B_3\beta_{hm}\lambda_1^*\left(f_1\left(\lambda_2^*+f_2\right)+f_5\lambda_2^*+f_3\right) \\ +\left(\left(\lambda_1^*+\mu_h\right)f_4-\theta_1\delta_h\eta_h\lambda_1^*\right)\lambda_2^*\pi\mu_h N_h\end{matrix}}{N_h\left(\lambda_2^*+f_2\right)\left(\left(\lambda_1^*+\mu_h\right)f_4-\theta_1\delta_h\eta_h\lambda_1^*\right)\left(\mu_h+\eta_h+a\right)} \qquad (4.26)$$

Then we substitute (4.24) and (4.25) into (4.26), to get

$$\lambda_3^*\left(A\lambda_3^{*2}+B\lambda_3^*+C\right)=0$$

where

$$\begin{aligned}A=&-\varphi f_2N_h(a+\eta_h+\mu_h)(\mu_m+u_4)\Big(\varphi f_4\mu_h(\mu_m+u_4)+(1-u_2)kB_1\beta_{mh}(f_4-\delta_h\eta_h\theta_1)\big((1-u_5)\\&\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\big)\Big)-(1-u_2)kB_2N_h\beta_{mh}(a+\eta_h+\mu_h)\\&\left((1-u_5)\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\right)\Big(\varphi f_4\mu_h(\mu_m+u_4)+(1-u_2)\\&kB_1\beta_{mh}(f_4-\delta_h\eta_h\theta_1)\left((1-u_5)\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\right)\Big)\end{aligned}$$

$$\begin{aligned}B=&-\varphi^2f_2f_4N_h\mu_h(a+\eta_h+\mu_h)(\mu_m+u_4)^3+(1-u_2)k\pi\varphi B_2f_4N_h\beta_{mh}\mu_h^2(\mu_m+u_4)\big((1-u_5)\varphi\eta_A\\&-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\big)-(1-u_2)k\varphi B_2f_4N_h\beta_{mh}\mu_h(a+\eta_h+\mu_h)(\mu_m+u_4)^2\\&\left((1-u_5)\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\right)-\varphi f_2N_h(a+\eta_h+\mu_h)(\mu_m+u_4)^2\\&\Big(\varphi f_4\mu_h(\mu_m+u_4)+(1-u_2)kB_1\beta_{mh}(f_4-\delta_h\eta_h\theta_1)\big((1-u_5)\varphi\eta_A-\big(u_3+(1-u_5)\eta_A+\\&\mu_A\big)(\mu_m+u_4)\big)\Big)+(1-u_2)^2kB_1\beta_{mh}\left((1-u_5)\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\right)\\&\Big(\varphi B_3(f_1f_2+f_3)\beta_{hm}(\mu_m+u_4)+kB_2\beta_{mh}\big((1-u_2)B_3(f_1+f_5)\beta_{hm}+\pi N_h(f_4-\delta_h\eta_h\theta_1)\\&\mu_h\big)\left((1-u_5)\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\right)\Big)\end{aligned}$$

$$C=-\varphi^2f_2f_4N_h\mu_h(a+\eta_h+\mu_h)(\mu_m+u_4)^4+(1-u_2)^2k\varphi B_1B_3(f_1f_2+f_3)\beta_{hm}\beta_{mh}(\mu_m+u_4)^2$$

$$\begin{aligned}&\left((1-u_5)\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\right)+(1-u_2)k\pi\varphi B_2f_4N_h\beta_{mh}\mu_h^2(\mu_m+u_4)^2\\&\quad\left((1-u_5)\varphi\eta_A-\left(u_3+(1-u_5)\eta_A+\mu_A\right)(\mu_m+u_4)\right)\end{aligned}$$

Consequently $\lambda_3^*=0$ or $A\lambda_3^{*2}+B\lambda_3^*+C=0$

Hence the disease free equilibrium is obtained when $\lambda_3^* = 0$ and endemic when

$$A\lambda_3^{*2} + B\lambda_3^* + C = 0 \tag{4.27}$$

For disease free equilibrium $\lambda_3^* = 0$

It follows that from (4.20) $I_m^* = 0$ then from (4.10) and (4.11)

$\lambda_1^* = 0$ and $\lambda_2^* = 0$

Thus from (4.14) to (4.18) and from (4.21) to (4.23) we have

$$S_{h_1}^* = \frac{(1-\pi)N_h(\mu_h + u_1\theta_2) + u_1\theta_2\pi N_h}{(\mu_h + u_1\theta_2)}, S_{h_2}^* = \frac{\pi\mu_h N_h}{\mu_h + u_1\theta_2}, I_h^* = 0, T_h^* = 0, R_h^* = 0,$$

$$A_m^* = \frac{kN_h q}{\varphi((1-u_5)\eta_A)}, S_m^* = \frac{kN_h q}{\varphi(\mu_m + u_4)} \text{ and } I_m^* = 0.$$

where $q = -((\mu_A + u_3 + (1-u_5)\eta_A)(\mu_m + u_4) - \varphi(1-u_5)\eta_A)$

Therefore the Disease Free Equilibrium (DFE) denoted by E_0 of the system (4.1) is given by $E_0 = (S_{h_1}(t), S_{h_2}(t), 0,0,0, A_m(t), S_m(t), 0) =$

$$\left(\frac{(1-\pi)N_h(\mu_h + u_1\theta_2) + u_1\theta_2\pi N_h}{\mu_h + u_1\theta_2}, \frac{\pi\mu_h N_h}{\mu_h + u_1\theta_2}, 0,0,0, \frac{qkN_h}{\varphi(1-u_5)\eta_A}, \frac{qkN_h}{\varphi(\mu_m + u_4)}, 0\right) \tag{4.28}$$

4.5 The Reproduction Number with control R_c

The reproduction number with control R_c is the threshold which is used for predicting outbreaks and evaluating control strategies that would reduce the spread of the disease. The dynamics of dengue fever disease is determined by the reproduction number with control R_c which is a key concept and is defined as the average number of secondary infection arising from a single infected individual introduced into the susceptible class during its entire infectious period in a totally susceptible population in which control strategies are included, for if R_c<1 the result is disease free-equilibrium and if R_c>1 endemic equilibrium point exists (Driessche and Watmough, 2002, Lashari *et al.*, 2013).

The model system of equations (4.1) will be analysed qualitatively to get a better understanding of the effects of controls, careful and Careless human Susceptibles of Dengue fever disease. The reproduction number with control of the model (4.1) R_c is calculated by using the next generation matrix of an ODE (Driessche and Watmough, 2002). Using the approach of Driessche and Watmough, (2002). R_c is obtained by taking the largest (dominant) Eigen value (spectral radius) of

$$\left[\frac{\partial F_i(E_0)}{\partial X_j}\right]\left[\frac{\partial V_i(E_0)}{\partial X_j}\right]^{-1},$$

where, F_i is the rate of appearance of new infection in compartment i, V_i^+ is the transfer of individuals out of the compartment i by all other means and E_0 is the disease free equilibrium. Therefore

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \left((1-u_2)B_1 S_{h_1} + (1-u_2)B_2 S_{h_2}\right)\beta_{mh}\dfrac{I_m}{N_h} \\ (1-u_2)B_3\beta_{hm}\dfrac{I_h}{N_h}S_m \end{bmatrix}$$

Using the linearization method, the associated matrix at DFE is given by

$$\mathbf{F} = \begin{pmatrix} \dfrac{\partial F_1}{\partial I_h}(E_0) & \dfrac{\partial F_1}{\partial I_m}(E_0) \\ \dfrac{\partial F_2}{\partial I_h}(E_0) & \dfrac{\partial F_2}{\partial I_m}(E_0) \end{pmatrix}.$$

This implies that

$$\mathbf{F} = \begin{pmatrix} 0 & \dfrac{\left((1-u_2)B_1S_{h_1} + (1-u_2)B_2S_{h_2}\right)\beta_{mh}}{N_h} \\ (1-u_2)B_3\beta_{hm}\dfrac{S_m}{N_h} & 0 \end{pmatrix}$$

with

$$S_{h_1} = \frac{(1-\pi)N_h(\mu_h + u_1\theta_2) + u_1\theta_2\pi N_h}{\mu_h + u_1\theta_2},\ S_{h_2} = \frac{\pi\mu_h N_h}{\mu_h + u_1\theta_2}\ \ S_m = \frac{qkN_h}{\varphi(\mu_m + u_4)}$$ we have

$$\mathbf{F} = \begin{pmatrix} 0 & \begin{pmatrix} \dfrac{(1-u_2)B_1(1-\pi)(\mu_h + u_1\theta_2) + u_1\theta_2\pi}{\mu_h + u_1\theta_2} + \\ (1-u_2)B_2\dfrac{\pi\mu_h}{\mu_h + u_1\theta_2} \end{pmatrix}\beta_{mh} \\ \dfrac{kq(1-u_2)B_3\beta_{hm}}{\varphi(\mu_m + u_4)} & 0 \end{pmatrix}$$

The transfer of individuals out of the compartment i is given by

$$\mathbf{V}_i = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} (\mu_h + \eta_h + a)I_h \\ (\mu_m + u_4)I_m \end{bmatrix}$$

Using the linearization method, the associated matrix at DFE is given by

$$\mathbf{V} = \begin{pmatrix} \dfrac{\partial V_1}{\partial I_h}(E_0) & \dfrac{\partial V_1}{\partial I_m}(E_0) \\ \dfrac{\partial V_2}{\partial I_h}(E_0) & \dfrac{\partial V_2}{\partial I_m}(E_0) \end{pmatrix}$$

This gives

$$\mathbf{V} = \begin{pmatrix} \mu_h + \eta_h + a & 0 \\ 0 & \mu_m + u_4 \end{pmatrix}$$

with

$$\mathbf{V}^{-1} = \begin{pmatrix} \dfrac{1}{\mu_h + \eta_h + a} & 0 \\ 0 & \dfrac{1}{\mu_m + u_4} \end{pmatrix}$$

Therefore

$$\mathbf{FV}^{-1} = \begin{pmatrix} 0 & \left(\frac{(1-u_2)B_1(1-\pi)(\mu_h + u_1\theta_2) + u_1\theta_2\pi}{\mu_h + u_1\theta_2} + (1-u_2)B_2 \frac{\pi\mu_h}{\mu_h + u_1\theta_2} \right)\beta_{mh} \\ \frac{kq(1-u_2)B_3\beta_{hm}}{\varphi(\mu_m + u_4)} & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\mu_h + \eta_h + a} & 0 \\ 0 & \frac{1}{\mu_m + u_4} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \left(\frac{(1-u_2)B_1(1-\pi)(\mu_h + u_1\theta_2) + u_1\theta_2\pi}{\mu_h + u_1\theta_2} + (1-u_2)B_2 \frac{\pi\mu_h}{\mu_h + u_1\theta_2} \right) \frac{\beta_{mh}}{\mu_m + u_4} \\ \frac{kq(1-u_2)B_3\beta_{hm}}{\varphi(\mu_m + u_4)(\mu_h + \eta_h + a)} & 0 \end{pmatrix} \quad (4.29)$$

The eigenvalues of the equation (4.29) are given by

$$\lambda^2 = \left(\frac{\left((1-u_2)B_1(1-\pi)(\mu_h + u_1\theta_2) + u_1\theta_2\pi + (1-u_2)B_2\pi\mu_h\right)kq(1-u_2)B_3\beta_{mh}\beta_{hm}}{\varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)^2(\mu_h + \eta_h + a)} \right)$$

which are

$$\lambda_1 = \sqrt{\left(\frac{\left((1-u_2)B_1(1-\pi)(\mu_h + u_1\theta_2) + u_1\theta_2\pi + (1-u_2)B_2\pi\mu_h\right)kq(1-u_2)B_3\beta_{mh}\beta_{hm}}{\varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)^2(\mu_h + \eta_h + a)} \right)}$$

or

$$\lambda_2 = -\sqrt{\left(\frac{\left((1-u_2)B_1(1-\pi)(\mu_h + u_1\theta_2) + u_1\theta_2\pi + (1-u_2)B_2\pi\mu_h\right)kq(1-u_2)B_3\beta_{mh}\beta_{hm}}{\varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)^2(\mu_h + \eta_h + a)} \right)}$$

It follows that the Reproductive number with control, which is given by the largest Eigen value for model system (4.1) denoted by R_c is given by

$$R_c = \sqrt{\left(\frac{\left((1-u_2)B_1(1-\pi)(\mu_h + u_1\theta_2) + u_1\theta_2\pi + (1-u_2)B_2\pi\mu_h\right)kq(1-u_2)B_3\beta_{mh}\beta_{hm}}{\varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)^2(\mu_h + \eta_h + a)} \right)}$$

But $q=-\left(\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)-\varphi(1-u_5)\eta_A\right)$.

Then

$$R_c=\sqrt{\left(\frac{-kB_3\beta_{mh}\beta_{hm}\left(\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)-\varphi(1-u_5)\eta_A\right)\times\left(\left(1-u_2\right)B_1(1-\pi)(\mu_h+u_1\theta_2)+u_1\theta_2\pi+(1-u_2)B_2\pi\mu_h\right)(1-u_2)}{\varphi(\mu_h+u_1\theta_2)(\mu_m+u_4)^2(\mu_h+\eta_h+a)}\right)}$$

or

$$R_c=\sqrt{\left(\frac{-kB_3\beta_{mh}\beta_{hm}t}{\varphi(\mu_h+u_1\theta_2)(\mu_m+u_4)^2(\mu_h+\eta_h+a)}\right)} \qquad (4.30)$$

$$t=\left(\left(\mu_A+u_3+(1-u_5)\eta_A\right)(\mu_m+u_4)-\varphi(1-u_5)\eta_A\right)\left((1-u_2)B_1(1-\pi)\right.$$

$$\left.(\mu_h+u_1\theta_2)+u_1\theta_2\pi+(1-u_2)B_2\pi\mu_h\right)(1-u_2)$$

Model System (4.1) has infection-free equilibrium E_0 if $R_c<1$, otherwise endemic equilibrium exists. Generally, the larger the value of R_c, the more severe, and possibly widespread the epidemic will be (Rodrigues *et al*., 2013).

4.6 Local Stability of Disease Free Equilibrium Point

The disease free of the non- linear model system (4.1) is given by

$$E_0=\left(S_{h_1}(t),S_{h_2}(t),0,0,0,A_m(t),S_m(t),0\right)=$$

$$\left(\frac{(1-\pi)N_h(\mu_h+u_1\theta_2)+u_1\theta_2\pi N_h}{\mu_h+u_1\theta_2},\frac{\pi\mu_hN_h}{\mu_h+u_1\theta_2},0,0,0,\frac{qkN_h}{\varphi(1-u_5)\eta_A},\frac{qkN_h}{\varphi(\mu_m+u_4)},0\right)$$

from (4.28)

Theorem 4.1:

The disease free equilibrium of the Dengue fever disease with optimal control model system (4.1) is locally asymptotically stable if $R_c < 1$ and is unstable if $R_c > 1$, that is Dengue disease can die out from the community if $\mathrm{R}_c < 1$, and will persist in the community if $R_c > 1$. Local stability of DFE point is determined by the variational matrix $\mathbf{J}_{E_0}$ of the non- linear system (4.1) corresponding to E_0 as follows,

Let

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - (1-u_2)B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + u_1\theta_2 S_{h_2} = Q_1\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - (1-u_2)B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - u_1\theta_2 S_{h_2} = Q_2\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dI_h}{dt} = \left((1-u_2)B_1 S_{h_1} + (1-u_2)B_2 S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h)I_h - aI_h = Q_3\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dT_h}{dt} = \eta_h I_h - (\mu_h + \delta_h)T_h = Q_4\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right) \tag{4.31}$$

$$\frac{dR_h}{dt} = \delta_h T_h - (\mu_h + \theta_1)R_h = Q_5\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dA_m}{dt} = \varphi\left(1 - \frac{A_m}{kN_h}\right)(S_m + I_m) - \mu_A A_m - u_3 A_m - (1-u_5)\eta_A A_m = Q_6\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dS_m}{dt} = (1-u_5)\eta_A A_m - \left((1-u_2)B_3\beta_{hm}\frac{I_h}{N_h} + \mu_m\right)S_m - u_4 S_m = Q_7\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

$$\frac{dI_m}{dt} = (1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}S_m - (\mu_m + u_4)I_m = Q_8\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$$

It follows that,

$$
\mathbf{J}_{E_0}=\begin{bmatrix}
\frac{\partial Q_1}{\partial S_{h_1}}(E_0) & \frac{\partial Q_1}{\partial S_{h_2}}(E_0) & \frac{\partial Q_1}{\partial I_h}(E_0) & \frac{\partial Q_1}{\partial T_h}(E_0) & \frac{\partial Q_1}{\partial R_h}(E_0) & \frac{\partial Q_1}{\partial A_m}(E_0) & \frac{\partial Q_1}{\partial S_m}(E_0) & \frac{\partial Q_1}{\partial I_m}(E_0) \\
\frac{\partial Q_2}{\partial S_{h_1}}(E_0) & \frac{\partial Q_2}{\partial S_{h_2}}(E_0) & \frac{\partial Q_2}{\partial I_h}(E_0) & \frac{\partial Q_2}{\partial T_h}(E_0) & \frac{\partial Q_2}{\partial R_h}(E_0) & \frac{\partial Q_2}{\partial A_m}(E_0) & \frac{\partial Q_2}{\partial S_m}(E_0) & \frac{\partial Q_2}{\partial I_m}(E_0) \\
\frac{\partial Q_3}{\partial S_{h_1}}(E_0) & \frac{\partial Q_3}{\partial S_{h_2}}(E_0) & \frac{\partial Q_3}{\partial I_h}(E_0) & \frac{\partial Q_3}{\partial T_h}(E_0) & \frac{\partial Q_3}{\partial R_h}(E_0) & \frac{\partial Q_3}{\partial A_m}(E_0) & \frac{\partial Q_3}{\partial S_m}(E_0) & \frac{\partial Q_3}{\partial I_m}(E_0) \\
\frac{\partial Q_4}{\partial S_{h_1}}(E_0) & \frac{\partial Q_4}{\partial S_{h_2}}(E_0) & \frac{\partial Q_4}{\partial I_h}(E_0) & \frac{\partial Q_4}{\partial T_h}(E_0) & \frac{\partial Q_4}{\partial R_h}(E_0) & \frac{\partial Q_4}{\partial A_m}(E_0) & \frac{\partial Q_4}{\partial S_m}(E_0) & \frac{\partial Q_4}{\partial I_m}(E_0) \\
\frac{\partial Q_5}{\partial S_{h_1}}(E_0) & \frac{\partial Q_5}{\partial S_{h_2}}(E_0) & \frac{\partial Q_5}{\partial I_h}(E_0) & \frac{\partial Q_5}{\partial T_h}(E_0) & \frac{\partial Q_5}{\partial R_h}(E_0) & \frac{\partial Q_5}{\partial A_m}(E_0) & \frac{\partial Q_5}{\partial S_m}(E_0) & \frac{\partial Q_5}{\partial I_m}(E_0) \\
\frac{\partial Q_6}{\partial S_{h_1}}(E_0) & \frac{\partial Q_6}{\partial S_{h_2}}(E_0) & \frac{\partial Q_6}{\partial I_h}(E_0) & \frac{\partial Q_6}{\partial T_h}(E_0) & \frac{\partial Q_6}{\partial R_h}(E_0) & \frac{\partial Q_6}{\partial A_m}(E_0) & \frac{\partial Q_6}{\partial S_m}(E_0) & \frac{\partial Q_6}{\partial I_m}(E_0) \\
\frac{\partial Q_7}{\partial S_{h_1}}(E_0) & \frac{\partial Q_7}{\partial S_{h_2}}(E_0) & \frac{\partial Q_7}{\partial I_h}(E_0) & \frac{\partial Q_7}{\partial T_h}(E_0) & \frac{\partial Q_7}{\partial R_h}(E_0) & \frac{\partial Q_7}{\partial A_m}(E_0) & \frac{\partial Q_7}{\partial S_m}(E_0) & \frac{\partial Q_7}{\partial I_m}(E_0) \\
\frac{\partial Q_8}{\partial S_{h_1}}(E_0) & \frac{\partial Q_8}{\partial S_{h_2}}(E_0) & \frac{\partial Q_8}{\partial I_h}(E_0) & \frac{\partial Q_8}{\partial T_h}(E_0) & \frac{\partial Q_8}{\partial R_h}(E_0) & \frac{\partial Q_8}{\partial A_m}(E_0) & \frac{\partial Q_8}{\partial S_m}(E_0) & \frac{\partial Q_8}{\partial I_m}(E_0)
\end{bmatrix}
$$

Hence the variational matrix of the non-linear model system (4.31) is obtained as

$$
\mathbf{J}_{E_0}=\begin{bmatrix}
-\mu_h & u_1\theta_2 & 0 & 0 & \theta_1 & 0 & 0 & f \\
0 & -u_1\theta_2-\mu_h & 0 & 0 & 0 & 0 & 0 & y \\
0 & 0 & -a-\eta_h-\mu_h & 0 & 0 & 0 & 0 & j \\
0 & 0 & \eta_h & -\delta_h-\mu_h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_h & -\theta_1-\mu_h & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t & \varphi[1-\frac{q}{\varphi\eta_A(1-u_5)}] & g \\
0 & 0 & -\frac{(1-u_2)kqB_3\beta_{hm}}{\varphi(\mu_m+u_4)} & 0 & 0 & \eta_A(1-u_5) & -\mu_m-u_4 & 0 \\
0 & 0 & \frac{(1-u_2)kqB_3\beta_{hm}}{\varphi(\mu_m+u_4)} & 0 & 0 & 0 & 0 & -\mu_m-u_4
\end{bmatrix} \quad (4.32)
$$

where

$$
t=-\mu_A-u_3-\eta_A(1-u_5)-\frac{q\varphi'\left(1-\frac{q}{\varphi\eta_A(1-u_5)}\right)}{\varphi(\mu_m+u_4)},
$$

$$
f=-\frac{(1-u_2)B_1\beta_{mh}\left(u_1\theta_2-(-1+\pi)\mu_h\right)}{u_1\theta_2+\mu_h},
$$

$$
y=-\frac{(1-u_2)\pi B_2\beta_{mh}\mu_h}{u_1\theta_2+\mu_h}, \qquad g=\varphi\left(1-\frac{q}{\varphi\eta_A(1-u_5)}\right),
$$

$$j = \frac{(1-u_2)\beta_{mh}\left(\pi B_2\mu_h + B_1\left(u_1\theta_2 - (-1+\pi)\mu_h\right)\right)}{u_1\theta_2 + \mu_h}.$$

Therefore the stability of the disease free equilibrium point can be clarified by studying the behaviour of $\mathbf{J}_{E_0}$ in which for local stability of DFE we seek for its all eigenvalues to have negative real parts. It follows that, the characteristic function of the matrix (4.32) with λ being the eigenvalues of the Jacobian matrix, by using Mathematica software the Jacobian matrix has the following eigenvalues:

$\lambda_1 = -\mu_h$, $\lambda_2 = -u_1\theta_2 - \mu_h$

The other eigenvalues are given as

$$\lambda_3 = \frac{1}{2}\left(-a - \eta_h - \mu_h - \mu_m - u_4 - \frac{\sqrt{\sigma}}{\sqrt{\varphi}\sqrt{u_1\theta_2 + \mu_h}\sqrt{\mu_m + u_4}}\right)$$ when $\sqrt{\sigma}$ is not a real

number

$\lambda_4 = -\delta_h - \mu_h$, $\lambda_5 = -\theta_1 - \mu_h$

$$\lambda_6 = \frac{1}{2}\left(-a - \eta_h - \mu_h - \mu_m - u_4 + \frac{\sqrt{\sigma}}{\sqrt{\varphi}\sqrt{u_1\theta_2 + \mu_h}\sqrt{\mu_m + u_4}}\right)$$ when $\sqrt{\sigma}$ is not real number

$$\lambda_7 = -\frac{1}{2\varphi(u_1\theta_2 + \mu_h)(\mu_m + u_4)}\left((u_1\theta_2 + \mu_h)\left(\varphi(\mu_m + u_4)(\mu_A + \mu_m + u_3 + u_4 + \eta_A(1-u_5)) + q\varphi'\left(1 - \frac{q}{\varphi\eta_A(1-u_5)}\right)\right) + \sqrt{\alpha}\right)$$

When $\sqrt{\alpha}$ is not a real number and finally

$$\lambda_8 = -\frac{1}{2\varphi(u_1\theta_2 + \mu_h)(\mu_m + u_4)}\left((u_1\theta_2 + \mu_h)\left(\varphi(\mu_m + u_4)(\mu_A + \mu_m + u_3 + u_4 + \eta_A(1-u_5)) + q\varphi'\left(1 - \frac{q}{\varphi\eta_A u_5}\right)\right) - \sqrt{\alpha}\right)$$

when $\sqrt{\alpha}$ is not a real number.

where,

$$\sigma = 4\left(1-u_2\right)^2 k\pi q B_2 B_3 \beta_{hm}\beta_{mh}\mu_h + 4\left(1-u_2\right)^2 kqB_1B_3\beta_{hm}\beta_{mh}\left(u_1\theta_2 - \left(-1+\pi\right)\mu_h\right)$$

$$+\varphi\left(u_1\theta_2+\mu_h\right)\left(a+\eta_h+\mu_h-\mu_m-u_4\right)^2\left(\mu_m+u_4\right)$$

$$\alpha = \Bigg(\left(u_1\theta_2+\mu_h\right)^2\Bigg(\varphi^2\mu_A^2\left(\mu_m+u_4\right)^2+\varphi^2\left(\mu_m+u_4\right)^2\Bigg(\left(\mu_m-u_3+u_4-\eta_A\left(1-u_5\right)\right)^2+4\eta_A\left(1-u_5\right)\varphi$$

$$\left(1-\frac{q}{\varphi\eta_A\left(1-u_5\right)}\right)\Bigg)\Bigg)-2q\varphi\left(\mu_m+u_4\right)\left(\mu_m-u_3+u_4-\eta_A\left(1-u_5\right)\right)\varphi'\left(1-\frac{q}{\varphi\eta_A\left(1-u_5\right)}\right)+q^2\varphi'$$

$$\left(1-\frac{q}{\varphi\eta_A\left(1-u_5\right)}\right)^2+2\varphi\mu_A\left(\mu_m+u_4\right)\left(-\varphi\left(\mu_m+u_4\right)\left(\mu_m-u_3+u_4-\eta_A\left(1-u_5\right)\right)+q\varphi'\left(1-\frac{q}{\varphi\eta_A\left(1-u_5\right)}\right)\right)\Bigg)\Bigg)$$

Hence the system is stable since all the eight eigenvalues are negative. This implies that at $R_c < 1$ the Disease Free Equilibrium point is locally asymptotically stable, that is Dengue infection can be eliminated from the population.

4.7 Global Stability of Disease Free Equilibrium Point

In this section, the global behaviour of the equilibria for system (4.1) is analysed. The following theorem provides the global property of the disease free equilibrium E_0 of the system. The results are obtained by means of Lyapunov function. In choosing the Lyapunov function we adopt the idea of Ozair *et al.,* (2013).

Theorem 4.2: If $R_c \leq 1$, then the infection-free equilibrium is globally asymptotically stable in the interior of Ω

Proof:

To establish the global stability of the disease-free equilibrium, the following Lyapunov function is constructed:

$$L(t) = -kB_3\beta_{hm}tI_h(t) + \varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)\mu_m I_m(t) \tag{4.33}$$

Calculating the time derivative of L along (4.33), we obtain

$$L^{'}(t) = -kB_3\beta_{hm}tI_h^{'}(t) + \varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)\mu_m I_m^{'}(t)$$

Then substituting $I_h^{'}(t)$ & $I_m^{'}(t)$ from system (4.1), we get

$$L^{'}(t) = -kB_3\beta_{hm}t\left(\left((1-u_2)B_1S_{h_1} + (1-u_2)B_2S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} - (\mu_h + \eta_h + a)I_h\right)$$

$$+\varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)\mu_m\left((1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}S_m - (\mu_m + u_4)I_m\right)$$

It follows that

$$L^{'}(t) = -kB_3\beta_{hm}t\left((1-u_2)B_1S_{h_1} + (1-u_2)B_2S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h} + kB_3\beta_{hm}t(\mu_h + \eta_h + a)I_h + \varphi(\mu_h + \eta_h$$

$$+a)(\mu_h + u_1\theta_2)(\mu_m + u_4)(1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}S_m - \varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)(\mu_m + u_4)^2 I_m$$

or

$$L^{'}(t) = \varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)(\mu_m + u_4)^2 I_m\left(\frac{-kB_3\beta_{hm}\beta_{mh}t\left((1-u_2)B_1S_{h_1} + (1-u_2)B_2S_{h_2}\right)}{N_h\varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)(\mu_m + u_4)^2} - 1\right)$$

$$+B_3\beta_{hm}(\mu_h + \eta_h + a)\left(ktI_h + \varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)(1-u_2)\frac{I_h}{N_h}S_m\right) \text{ Then}$$

$$L^{'}(t) = \varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)(\mu_m + u_4)^2 I_m\left(\frac{\left((1-u_2)B_1S_{h_1} + (1-u_2)B_2S_{h_2}\right)}{N_h}R_c^2 - 1\right)$$

$$-\frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)^2 R_c^2}\left(ktI_h + \varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)(1-u_2)\frac{I_h}{N_h}S_m\right).$$

Consequently

$$L^{'}(t) = \varphi(\mu_h + \eta_h + a)(\mu_h + u_1\theta_2)(\mu_m + u_4)^2 I_m\left(fR_c^2 - 1\right) - \frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi(\mu_h + u_1\theta_2)(\mu_m + u_4)^2 R_c^2}$$

$$\times\left(ktI_h+\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)\left(1-u_2\right)\frac{I_h}{N_h}S_m\right)$$

where

$$t=\left(\left(\mu_A+u_3+\left(1-u_5\right)\eta_A\right)\left(\mu_m+u_4\right)-\varphi\left(1-u_5\right)\eta_A\right)\left(\left(1-u_2\right)B_1\left(1-\pi\right)\left(\mu_h+u_1\theta_2\right)+u_1\theta_2\pi+\left(1-u_2\right)B_2\pi\mu_h\right)\left(1-u_2\right)$$

$$R_c^2=\frac{-kB_3\beta_{mh}\beta_{hm}t}{\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2\left(\mu_h+\eta_h+a\right)},\qquad f=\frac{\left(\left(1-u_2\right)B_1S_{h_1}+\left(1-u_2\right)B_2S_{h_2}\right)}{N_h}.$$

Therefore

$$L^{'}\left(t\right)=-\varphi\left(\mu_h+\eta_h+a\right)\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2I_m\left(\sqrt{f}R_c+1\right)\left(1-\sqrt{f}R_c\right)$$
$$-\frac{kB_3^2\beta_{hm}^2\beta_{mh}t}{\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2R_c^2}\left(ktI_h+\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)\left(1-u_2\right)\frac{I_h}{N_h}S_m\right)$$

Thus, $L'\left(t\right)$ is negative if $R_c\le 1$ and $L'=0$ if and only if $I_h=I_m=0$ is reduced to the DFE. Consequently, the largest compact invariant set in $\left\{\left(S_{h_1},S_{h_2},I_h,T_h,R_h,A_m,S_m,I_m\right)\in\Omega,\quad L'=0\right\}$ when $R_c\le 1$ is the singelton$\{\mathrm{E}_0\}$. Hence, by LaSalle's invariance principle implies that $"E_0"$ is globally asymptotically stable in Ω (LaSalle, 1976). This completes the proof.

4.8 Existence, Local and Global Asymptotic Stability of Endemic Equilibrium

Since we are dealing with presence of dengue fever disease in human population, we can reduce system (4.1) to a 4-dimensional system by eliminating T_h, R_h, A_m and S_m respectively, in the feasible region Ω. The values of S_m can be determined by setting $S_m=mN_h-I_m$ to obtain

$$\frac{dS_{h_1}}{dt}=\left(1-\pi\right)\mu_hN_h-\left(1-u_2\right)B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1}-\mu_hS_{h_1}+\theta_1R_h+u_1\theta_2S_{h_2}$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - (1-u_2) B_2 \beta_{mh} \frac{I_m}{N_h} S_{h_2} - \mu_h S_{h_2} - u_1\theta_2 S_{h_2} \tag{4.34}$$

$$\frac{dI_h}{dt} = \left((1-u_2) B_1 S_{h_1} + (1-u_2) B_2 S_{h_2}\right) \beta_{mh} \frac{I_m}{N_h} - (\mu_h + \eta_h + a) I_h$$

$$\frac{dI_m}{dt} = (1-u_2) B_3 \beta_{hm} \frac{I_h}{N_h} (mN_h - I_m) - (\mu_m + u_4) I_m$$

Then we set

$$\frac{dS_{h_1}}{dt} = \frac{dS_{h_1}}{dt} = \frac{dI_h}{dt} = \frac{dI_m}{dt} = 0.$$

Then the model of system (4.34) has a unique endemic equilibrium given by

$E^* = \left(S^*_{h_1}, S^*_{h_2}, I^*_h, I^*_m\right)$ in Ω, with

$$(1-\pi)\mu_h N_h - (1-u_2) B_1 \beta_{mh} \frac{I^*_m}{N_h} S^*_{h_1} - \mu_h S^*_{h_1} + \theta_1 R^*_h + u_1\theta_2 S^*_{h_2} = 0$$

$$\pi\mu_h N_h - (1-u_2) B_2 \beta_{mh} \frac{I_m}{N_h} S_{h_2} - \mu_h S_{h_2} - u_1\theta_2 S_{h_2} = 0 \tag{4.35}$$

$$\left((1-u_2) B_1 S^*_{h_1} + (1-u_2) B_2 S^*_{h_2}\right) \beta_{mh} \frac{I^*_m}{N_h} - (\mu_h + \eta_h + a) I^*_h = 0$$

$$(1-u_2) B_3 \beta_{hm} \frac{I^*_h}{N_h} \left(mN_h - I^*_m\right) - (\mu_m + u_4) I^*_m = 0.$$

4.8.1 Existence of Endemic Equilibrium

Existence of endemic equilibrium depends on the quadratic equation (4.27), that is, if it has positive roots. The sign of the roots depends on the sign of A, B and C. From (4.27) we have $A\lambda_3^{*2} + B\lambda_3^* + C = 0$

where

$$A = -\varphi f_2 N_h (a + \eta_h + \mu_h)(\mu_m + u_4)\left(\varphi f_4 \mu_h (\mu_m + u_4) + (1-u_2) k B_1 \beta_{mh} (f_4 - \delta_h \eta_h \theta_1)\right)$$

$$\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)\Big)-\left(1-u_2\right)kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)$$

$$\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)\Big(\varphi f_4\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}$$

$$\left(f_4-\delta_h\eta_h\theta_1\right)\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)\Big)$$

$$B=-\varphi^2 f_2 f_4N_h\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3+\left(1-u_2\right)k\pi\varphi B_2 f_4N_h\beta_{mh}\mu_h^2\left(\mu_m+u_4\right)\Big(\left(1-u_5\right)\varphi\eta_A-$$

$$\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big)-\left(1-u_2\right)k\varphi B_2 f_4N_h\beta_{mh}\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2$$

$$\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2$$

$$\Big(\varphi f_4\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)\Big(\left(1-u_5\right)\varphi\eta_A-\Big(u_3+\left(1-u_5\right)\eta_A+$$

$$\mu_A\Big)\left(\mu_m+u_4\right)\Big)\Big)+\left(1-u_2\right)^2 kB_1\beta_{mh}\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)$$

$$\Big(\varphi B_3\left(f_1f_2+f_3\right)\beta_{hm}\left(\mu_m+u_4\right)+kB_2\beta_{mh}\Big(\left(1-u_2\right)B_3\left(f_1+f_5\right)\beta_{hm}+\pi N_h$$

$$\left(f_4-\delta_h\eta_h\theta_1\right)\mu_h\Big)\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)\Big)$$

$$C=-\varphi^2 f_2 f_4N_h\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^4+\left(1-u_2\right)^2 k\varphi B_1B_3\left(f_1f_2+f_3\right)\beta_{hm}\beta_{mh}\left(\mu_m+u_4\right)^2$$
$$\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)+\left(1-u_2\right)k\pi\varphi B_2 f_4N_h\beta_{mh}\mu_h^2\left(\mu_m+u_4\right)^2$$
$$\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)$$

We express A, B and C in term of R_c as follows:

$$A=-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)\Big(\varphi f_4\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)$$

$$\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)\Big)-\left(1-u_2\right)kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)$$

$$\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)\Big(\varphi f_4\mu_h\mu_m+\left(1-u_2\right)kB_1\beta_{mh}$$

$$\left(f_4-\delta_h\eta_h\theta_1\right)\left(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\right)\Big)$$

or

$$A=-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)\left(\varphi f_4\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)q\right)-$$
$$\left(1-u_2\right)kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)q\left(\varphi f_4\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)q\right)$$

then

$$A=\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)\left(1-u_2\right)kB_1\beta_{mh}q\delta_h\eta_h\theta_1+\left(1-u_2\right)kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)\left(1-u_2\right)$$
$$qkB_1\beta_{mh}q\delta_h\eta_h\theta_1-k\left(1-u_2\right)^2B_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)qkB_1\beta_{mh}qf_4-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)$$
$$\left(\mu_m+u_4\right)\left(1-u_2\right)kB_1\beta_{mh}qf_4-\varphi f_2N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)\varphi f_4\mu_h\left(\mu_m+u_4\right)-$$
$$\left(1-u_2\right)kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)q\varphi f_4\mu_h\left(\mu_m+u_4\right)$$

It is follows that

$$A=t_1-\left(\left(1-u_2\right)^2kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)qkB_1\beta_{mh}qf_4+\left(1-u_2\right)kB_2N_h\beta_{mh}\left(a+\eta_h+\mu_h\right)q\varphi f_4\mu_h\right.$$
$$\left.\left(\mu_m+u_4\right)\right)-\left(a+\eta_h+\mu_h\right)f_2N_h\left(\mu_m+u_4\right)\varphi f_4\left(\varphi\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}q\right)$$

Then

$$A=t_1+\left(a+\eta_h+\mu_h\right)f_2\varphi N_h\left(\mu_m+u_4\right)f_4\left(\varphi\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}q\right)$$
$$\times\left(\frac{-\left(1-u_2\right)\beta_{mh}kB_2N_hq\left(\left(1-u_2\right)kB_1\beta_{mh}q+\varphi\mu_h\left(\mu_m+u_4\right)\right)}{f_2N_h\left(\mu_m+u_4\right)\varphi\left(\varphi\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}q\right)}-1\right)$$

or

$$A=t1+t2\left(\frac{-\beta_{mh}kN_hq\left(\left(1-u_2\right)B_2\varphi_4\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)^2B_2kB_1\beta_{mh}q\right)}{\varphi f_2N_h\left(\mu_m+u_4\right)\left(\varphi\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}q\right)}-1\right)$$ implying

that

$$A=t1+t2\left(\frac{N_hqf_4\left(1-u_2\right)B_2\left(\varphi\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}q\right)\left(\mu_h+\eta_h+a\right)\left(\mu_h+\theta_2\right)\left(\mu_m+u_4\right)}{f_2N_hf_4\left(\varphi\mu_h\left(\mu_m+u_4\right)+\left(1-u_2\right)kB_1\beta_{mh}q\right)\left(1-u_2\right)B_3\beta_{hm}t}R_c^2-1\right)$$

Consequently $A = t_1 + t_2\left(t_3 R_c^2 - 1\right)$

Therefore $A = t_1 + t_2\left(\sqrt{t_3}R_c + 1\right)\left(\sqrt{t_3}R_c - 1\right)$

in case of B we have

$$B = -\varphi^2 f_2 f_4 N_h \mu_h \left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3 + \left(1-u_2\right)k\pi\varphi B_2 f_4 N_h \beta_{mh}\mu_h^2\left(\mu_m+u_4\right)\Big(\left(1-u_5\right)\varphi\eta_A - \left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big) - \left(1-u_2\right)k\varphi B_2 f_4 N_h \beta_{mh}\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2 \Big(\left(1-u_5\right)\varphi\eta_A - \left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big) - \varphi f_2 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2 \left(\varphi f_4 \mu_h\left(\mu_m+u_4\right) + \left(1-u_2\right)kB_1\beta_{mh}\left(f_4-\delta_h\eta_h\theta_1\right)\Big(\left(1-u_5\right)\varphi\eta_A - \left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big)\right) + \left(1-u_2\right)kB_1\beta_{mh}\Big(\left(1-u_5\right)\varphi\eta_A - \left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big)\Big(\left(1-u_2\right)\varphi B_3\left(f_1 f_2+f_3\right)\beta_{hm}\left(\mu_m+u_4\right) + \left(1-u_2\right)kB_2\beta_{mh}\Big(\left(1-u_2\right)B_3\left(f_1+f_5\right)\beta_{hm} + \pi N_h\left(f_4 - \delta_h\eta_h\theta_1\right)\mu_h\Big)\Big(\left(1-u_5\right)\varphi\eta_A - \left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big)\Big)$$

or

$$B = \left(1-u_2\right)k\pi\varphi B_2 f_4 N_h\beta_{mh}\mu_h^2\left(\mu_m+u_4\right)q + \varphi f_2 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2\left(1-u_2\right)kB_1\beta_{mh}\delta_h\eta_h\theta_1 q + \left(1-u_2\right)^2 kB_1\beta_{mh}q\varphi B_3\left(f_1 f_2+f_3\right)\beta_{hm}\left(\mu_m+u_4\right) + \left(1-u_2\right)^3 kB_1\beta_{mh}qkB_2\beta_{mh}qB_3\beta_{hm}\left(f_1+f_5\right) + \left(1-u_2\right)^2 kB_1\beta_{mh}qkB_2\beta_{mh}q\pi N_h\mu_h f_4 - \varphi^2 f_2 f_4 N_h\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3 - \left(1-u_2\right) k\varphi B_2 f_4 N_h\beta_{mh}\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2 q - \varphi f_2 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2\varphi f_4\mu_h \left(\mu_m+u_4\right) - \varphi f_2 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2\left(1-u_2\right)kB_1\beta_{mh}qf - \left(1-u_2\right)^2 kB_1\beta_{mh}q kB_2\beta_{mh}q\pi N_h\delta_h\eta_h\theta_1\mu_h$$

It follows that

$$B = t_4 + t_5 - (\left(1-u_2\right)k\varphi B_2 f_4 N_h\beta_{mh}\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2 q + \varphi f_2 N_h\left(a+\eta_h+\mu_h\right)$$

$$\left(\mu_m+u_4\right)^2\left(1-u_2\right)kB_1\beta_{mh}qf_4+\left(1-u_2\right)^2 kB_1\beta_{mh}qkB_2\beta_{mh}q\pi N_h\delta_h\eta_h\theta_1\mu_h\Big)-$$

$$2\varphi^2 f_2 f_4 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3\mu_h \quad \text{or}$$

$$B=t_4+t_5-k\beta_{mh}N_hq(\varphi\left(1-u_2\right)B_2f_4\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2+\varphi f_2\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2$$
$$\left(1-u_2\right)B_1f_4+\left(1-u_2\right)^2 B_1qkB_2\beta_{mh}\pi\delta_h\eta_h\theta_1\mu_h\Big)-2\varphi^2 f_2 f_4 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3\mu_h$$

then

$$B=t_4+t_5+2\varphi^2 f_2 f_4 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3\mu_h$$

$$\times\left(\frac{-k\beta_{mh}N_hq\Big(\left(1-u_2\right)\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2 f_4\varphi\left(B_2\mu_h+f_2B_1\right)+\left(1-u_2\right)^2 B_1qkB_2\beta_{mh}\pi\delta_h\eta_h\theta_1\mu_h\Big)}{2\varphi^2 f_2 f_4 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3\mu_h}-1\right)$$

or

$$B=t_4+t_5+2\varphi^2 f_2 f_4 N_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^3\mu_h$$

$$\times\left(\frac{N_hq\begin{pmatrix}\varphi\left(1-u_2\right)B_2f_4\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^2+\varphi f_2\left(a+\eta_h+\mu_h\right)\\ \left(\mu_m+u_4\right)^2\left(1-u_2\right)B_1f_4+\left(1-u_2\right)^2 B_1qkB_2\beta_{mh}\pi\delta_h\eta_h\theta_1\mu_h\end{pmatrix}\left(\mu_h+\theta_2\right)}{2\varphi f_2 f_4 N_h\left(\mu_m+u_4\right)\mu_h\left(1-u_2\right)B_3\beta_{hm}t}R_c^2-1\right)$$

Consequently $B=t_4+t_5+t_6\left(t_7R_c^2-1\right)$

For the case of C is

$$C=-\varphi^2 f_2 f_4 N_h\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^4+\left(1-u_2\right)^2 k\varphi B_1B_3\left(f_1f_2+f_3\right)\beta_{hm}\beta_{mh}\left(\mu_m+u_4\right)^2$$
$$\Big(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big)+\left(1-u_2\right)k\pi\varphi B_2f_4N_h\beta_{mh}\mu_h^2$$

$$\left(\mu_m+u_4\right)^2\Big(\left(1-u_5\right)\varphi\eta_A-\left(u_3+\left(1-u_5\right)\eta_A+\mu_A\right)\left(\mu_m+u_4\right)\Big) \quad \text{then}$$

$$C=-\varphi^2 f_2 f_4 N_h\mu_h\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)^4+\left(1-u_2\right)^2 k\varphi B_1B_3\left(f_1f_2+f_3\right)\beta_{hm}\beta_{mh}$$

$$\left(\mu_m+u_4\right)^2 q+\left(1-u_2\right)k\pi\varphi B_2f_4N_h\beta_{mh}\mu_h^2\left(\mu_m+u_4\right)^2 q \quad \text{or}$$

$$C = (1-u_2)^2 k\varphi B_1 B_3 (f_1 f_2 + f_3)\beta_{hm}\beta_{mh}(\mu_m + u_4)^2 q + (1-u_2)k\pi\varphi B_2 f_4 N_h \beta_{mh}\mu_h^2$$

$$(\mu_m + u_4)^2 q + \frac{\varphi^2 f_2 f_4 N_h \mu_h (\mu_m + u_4)^2 (1-u_2) k\beta_{mh} B_3 \beta_{hm} t}{(\mu_h + \theta_2)\varphi R_c^2}$$

It follows that

$$C = k\beta_{mh}\varphi(\mu_m + u_4)^2 \, f_4 \mu_h N_h t8 \left(\frac{(1-u_2)^2 kB_3 \beta_{hm}\beta_{mh}\varphi B_1 (f_1 f_2 + f_3)(\mu_m + u_4)^2 q}{k\beta_{mh}\varphi \mu_m^2 f_4 \mu_h N_h t8} + 1 \right)$$

or

$$C = k\beta_{mh}\varphi(\mu_m + u_4)^2 \, f_4 \mu_h N_h t_8 \left(-\frac{(1-u_2)\varphi B_1 (f_1 f_2 + f_3)(\mu_m + u_4)^2 q (\mu_h + \theta_2)(\mu_h + \eta_h + a)}{k\beta_{mh} f_4 \mu_h N_h t_8 t} R_c^2 + 1 \right)$$

Consequently $C = t_9\left(-t_{10}R_c^2 + 1\right)$ or $C = t_9\left(1 - t_{10}R_c^2\right)$

Hence $C = t_9\left(1 + \sqrt{t_{10}}R_c\right)\left(1 - \sqrt{t_{10}}R_c\right)$

Where

$$t_1 = (a + \eta_h + \mu_h)(1-u_2)N_h B_1 \beta_{mh} qk\delta_h \eta_h \theta_1 \left(\varphi f_2 (\mu_m + u_4) + (1-u_2)B_2 \beta_{mh} kq\right)$$

$$t_2 = (a + \eta_h + \mu_h)\varphi f_2 N_h (\mu_m + u_4) f_4 \left(\varphi\mu_h (\mu_m + u_4) + (1-u_2)kB_1 \beta_{mh} q\right)$$

$$t_3 = \frac{q\left((1-u_2)B_2 \varphi_4 \mu_h (\mu_m + u_4) + (1-u_2)^2 B_2 kB_1 \beta_{mh} q\right)(\mu_h + \eta_h + a)(\mu_h + \theta_2)(\mu_m + u_4)}{f_2 \left(\varphi\mu_h (\mu_m + u_4) + (1-u_2)kB_1 \beta_{mh} q\right)(1-u_2)B_3 \beta_{hm} t}$$

$$t_4 = (1-u_2)k\pi\varphi B_2 f_4 N_h \beta_{mh}\mu_h^2 (\mu_m + u_4)q + \varphi f_2 N_h (a + \eta_h + \mu_h)(\mu_m + u_4)^2 (1-u_2)kB_1 \beta_{mh}\delta_h \eta_h \theta_1 q$$

$$t_5 = (1-u_2)^2 \beta_{mh} qkB_1 \left(\varphi B_3 (f_1 f_2 + f_3)\beta_{hm}(\mu_m + u_4) + kB_2 \beta_{mh} q\left((1-u_2)B_3 \beta_{hm}(f_1 + f_5) + \pi N_h \mu_h f_4\right)\right)$$

$$t_6 = 2\varphi^2 f_2 f_4 N_h (a + \eta_h + \mu_h)(\mu_m + u_4)^3 \mu_h$$

$$t_7 = \frac{q\begin{pmatrix}\varphi(1-u_2)B_2 f_4\mu_h(a+\eta_h+\mu_h)(\mu_m+u_4)^2+\varphi f_2(a+\eta_h+\mu_h)(\mu_m+u_4)^2 \\ (1-u_2)B_1 f_4+(1-u_2)^2 B_1 qkB_2\beta_{mh}\pi\delta_h\eta_h\theta_1\mu_h\end{pmatrix}(\mu_h+\theta_2)}{2\varphi f_2 f_4(\mu_m+u_4)\mu_h(1-u_2)B_3\beta_{hm}t}$$

$$t_8 = \left(\pi(1-u_2)B_2\mu_h q+\frac{(1-u_2)f_2B_3\beta_{hm}t}{(\mu_h+\theta_2)R_c^2}\right) \qquad t_9 = k\beta_{mh}\varphi(\mu_m+u_4)^2 f_4\mu_h N_h t_8$$

$$q=\big((1-u_5)\varphi\eta_A-(u_3+(1-u_5)\eta_A+\mu_A)(\mu_m+u_4)\big)$$

$$t_{10} = \frac{\varphi(1-u_2)B_1(f_1f_2+f_3)(\mu_m+u_4)^2 q(\mu_h+\theta_2)(\mu_h+\eta_h+a)}{k\beta_{mh}f_4\mu_h N_h t_8 t}$$

$$R_c = \sqrt{-\frac{(1-u_2)k\beta_{mh}B_3\beta_{hm}t}{(\mu_h+\eta_h+a)(\mu_m+u_4)^2(\mu_h+\theta_2)\varphi}}$$

$$f_1 = (1-\pi)\mu_h N_h(\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a) \quad f_2 = (\mu_h+\theta_2)$$

$$f_5 = \theta_1\delta_h\eta_h\pi\mu_h N_h \quad f_3 = (\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)\theta_2\pi\mu_h N_h$$

$$f_4 = (\mu_h+\theta_1)(\mu_h+\delta_h)(\mu_h+\eta_h+a)$$

It is observed that the coefficients A, B are non- negative if $R_c > 1$ and C is positive if $R_c < 1$, C is negative if $R_c > 1$ so that $\lambda_3^* = \dfrac{-B \pm \sqrt{B^2-4AC}}{2A}$ from (4.27)

Therefore the model has :

i) Unique endemic equilibrium if $R_c = 1$ implying that $C = 0$

ii) Unique endemic equilibrium if $B > 0$ and $C = 0$ or $B^2 - 4AC = 0$

iii) Two endemic equilibria if $R_c > 1$ and $C < 0$ implying that $4AC < 0$

iv) No endemic otherwise

It is clear from (i) and (ii) that the model has a unique endemic equilibrium. Further, in (iii) indicates that the model has two endemic equilibria.

4.8.2 Local Stability of the Endemic Equilibrium

In order to analyse the stability of the endemic equilibrium, the additive compound matrices approach is used, using the idea of Lee and Lashari, (2014).

If $\mathrm{R}_c > 1$, then the host-vector model equations (4.35) has a unique endemic equilibrium given by $E^* = \left(S_{h_1}^*, S_{h_2}^*, I_h^*, I_m^*\right)$ in Ω, with

$$S_{h_1}^* = \Bigg(\bigg((1-u_2)N_h\beta_{mh}(B_2(2u_1\theta_2+\mu_h)(a+\eta_h+\mu_h)(\mu_m+u_4)+\frac{kB_3\beta_{mh}\beta_{hm}t}{\varphi(\mu_h+u_1\theta_2)(\mu_m+u_4)R_c^2}B_1(u_1\theta_2+\mu_h)\bigg)$$

$$+(1-u_2)^2 m+B_3N_h\beta_{hm}\beta_{mh}\Big(N_h\mu_h\big(u_1(2B_2-B_1)\theta_2+((2-\pi)B_2)\mu_h-(1-\pi)B_1\big)+(2B_2-B_1)\theta_1(u_1\theta_2+\mu_h)$$

$$R_h^*\Big)+\sqrt{\Bigg((1-u_2)^2N_h^2\beta_{mh}^2\bigg(4B_1B_2\mu_h(a+\eta_h+\mu_h)(u_1\theta_2+\mu_h)\bigg(\frac{kB_3\beta_{mh}\beta_{hm}t}{\varphi(\mu_h+u_1\theta_2)R_c^2}+(1-u_2)m(\mu_m+u_4)+}$$

$$B_3\beta_{hm}(\mu_m+u_4)(N_h\mu_h+\theta_1R_h^*)\Big)+\Big(B_2\mu_h\big(-(1-u_2)m\pi-B_3N_h\beta_{hm}\mu_h+(a+\eta_h+\mu_h)(\mu_m+u_4)\big)+$$

$$B_1\Big((a+\eta_h+\mu_h)(u_1\theta_2+\mu_h)(\mu_m+u_4)-(1-u_2)m-B_3\beta_{hm}\big(N_h\mu_h(u_1\theta_2+(1-\pi)\mu_h)+\theta_1(u_1\theta_2+\mu_h)$$

$$R_h^*\big)\Big)\Big)^2\bigg)\Bigg)\Bigg)\Bigg/\Big(2(1-u_2)^2m+(B_2-B_1)B_3N_h\beta_{hm}\beta_{mh}\mu_h(u_1\theta_2+\mu_h)\Big)$$

$$S_{h_2}^* = \Bigg(\bigg((1-u_2)N_h\beta_2(a+\eta_h+\mu_h)\bigg(B_2\mu_h+\frac{kB_3\beta_{mh}\beta_{hm}t}{\varphi(\mu_h+u_1\theta_2)(\mu_m+u_4)^2R_c^2}B_1(u_1\theta_2+\mu_h)\bigg)(\mu_m+u_4)$$

$$+(1-u_2)^2m+\Bigg(N_h\mu_h\bigg(N_hB_3\beta_{hm}\beta_{mh}\pi B_2\mu_h+\frac{(\mu_h+\eta_h+a)\varphi(\mu_h+u_1\theta_2)(\mu_m+u_4)^2R_c^2}{kt}$$

$$N_hB_1(u_1\theta_2+(1-\pi)\mu_h)\bigg)+\frac{(\mu_h+\eta_h+a)\varphi(\mu_h+u_1\theta_2)(\mu_m+u_4)^2R_c^2}{kt}N_hB_1\theta_1(u_1\theta_2+\mu_h)R_h^*\Bigg)$$

$$+\sqrt{\Big((1-u_2)^2N_h^2\beta_{mh}^2\big(4B_1B_2\mu_h(a+\eta_h+\mu_h)(u_1\theta_2+\mu_h)\bigg(\frac{kB_3\beta_{mh}\beta_{hm}t}{\varphi(\mu_h+u_1\theta_2)R_c^2}+(1-u_2)m(\mu_m+u_4)}$$

$$+B_3\beta_{hm}\left(\mu_m+u_4\right)\left(N_h\mu_h+\theta_1R_h^*\right)\Big)+\Big(B_2\mu_h\left(-\left(1-u_2\right)m\pi-B_3N_h\beta_{hm}\mu_h+\left(a+\eta_h+\mu_h\right)\left(\mu_m+u_4\right)\right)$$

$$+B_1\Big(\left(a+\eta_h+\mu_h\right)\left(u_1\theta_2+\mu_h\right)\left(\mu_m+u_4\right)-\left(1-u_2\right)m-B_3\beta_{hm}\left(N_h\mu_h\left(u_1\theta_2+\left(1-\pi\right)\mu_h\right)\right.$$
$$\left.+\theta_1\left(u_1\theta_2+\mu_h\right)R_h^*\right)\Big)^2\Big)\Big)\Big)\Big)/\Big(2\left(1-u_2\right)^2m+\left(B_2-B_1\right)B_3N_h\beta_{hm}\beta_{mh}\mu_h\left(u_1\theta_2+\mu_h\right)\Big),$$

$$I_h^*=\frac{\left(\mu_m+u_4\right)\beta_{mh}kt}{\beta_{mh}kt\left(1-u_2\right)m+\left(\mu_h+\eta_h+a\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2R_c^2}$$

$$I_m^*=\left(1-u_2\right)N_h\beta_{mh}(a+\eta_h+\mu_h)\left(u_1B_1\theta_2+(B_1+B_2)\mu_h\right)(\mu_m+u_4)+\left(1-u_2\right)^2m+B_3N_h\beta_{hm}\beta_{mh}$$
$$\left(N_h\mu_h\left(\pi B_2\mu_h+B_1\left(u_1\theta_2+\left(1-\pi\right)\mu_h\right)\right)+B_1\theta_1\left(u_1\theta_2+\mu_h\right)R_h^*\right)+\surd\Big(\left(1-u_2\right)^2N_h^2\beta_{mh}^2$$
$$(4B_1B_2\mu_h\left(a+\eta_h+\mu_h\right)\left(u_1\theta_2+\mu_h\right)\left(\frac{kB_3\beta_{mh}\beta_{hm}t}{\varphi(\mu_h+u_1\theta_2)R_c^2}+\left(1-u_2\right)m\left(\mu_m+u_4\right)+B_3\beta_{hm}\right.$$

$$\left(\mu_m+u_4\right)\left(N_h\mu_h+\theta_1R_h^*\right))+(B_2\mu_h(-\left(1-u_2\right)m\pi-B_3N_h\beta_{hm}\mu_h+(a+\eta_h+\mu_h)\left(\mu_m+u_4\right)\Big)+$$
$$B_1((a+\eta_h+\mu_h)(u_1\theta_2+\mu_h)\left(\mu_m+u_4\right)-\left(1-u_2\right)m-B_3\beta_{hm}(N_h\mu_h\left(u_1\theta_2+\left(1-\pi\right)\mu_h\right)+$$
$$\left.\left.\theta_1\left(u_1\theta_2+\mu_h\right)R_h^*\right)\Big)\Big)^2\Big)\Big)\Big)/(2\left(1-u_2\right)^2B_1B_2\beta_{mh}^2((a+\eta_h+\mu_h)\left(\mu_m+u_4\right)\left(\sqrt{T}R_c+1\right)$$
$$\left(\sqrt{T}R_c-1\right)+\left(1-u_2\right)m))$$

where

$$T=\frac{\left(N_h\mu_h+\theta_1R_h^*\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)}{k\beta_{mh}t},$$

$$R_c^2=\frac{-kB_3\beta_{mh}\beta_{hm}t}{\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2\left(\mu_h+\eta_h+a\right)}$$

Local stability of the endemic equilibrium point is determined by the variational matrix $\mathbf{J}(E^*)$ of the non- linear system (4.34) corresponding to E^* as follows:

Let

$$\frac{dS_{h_1}}{dt}=\left(1-\pi\right)\mu_hN_h-\left(1-u_2\right)B_1\beta_{mh}\frac{I_m^*}{N_h}S_{h_1}^*-\mu_hS_{h_1}^*+\theta_1R_h^*+u_1\theta_2S_{h_2}^*=G_1\left(S_{h_1},S_{h_2},I_h,I_m\right)$$

$$\frac{dS_{h_2}}{dt}=\pi\mu_hN_h-\left(1-u_2\right)B_2\beta_{mh}\frac{I_m^*}{N_h}S_{h_2}^*-\mu_hS_{h_2}^*-u_1\theta_2S_{h_2}^*=G_2\left(S_{h_1},S_{h_2},I_h,I_m\right)\qquad(4.36)$$

$$\frac{dI_h}{dt}=\left(\left(1-u_2\right)B_1S_{h_1}^*+\left(1-u_2\right)B_2S_{h_2}^*\right)\beta_{mh}\frac{I_m^*}{N_h}-\left(\mu_h+\eta_h+a\right)I_h^*=G_3\left(S_{h_1},S_{h_2},I_h,I_m\right)$$

$$\frac{dI_m}{dt}=\left(1-u_2\right)B_3\beta_{hm}\frac{I_h^*}{N_h}\left(mN_h-I_m^*\right)-\left(\mu_m+u_4\right)I_m^*=G_4\left(S_{h_1},S_{h_2},I_h,I_m\right)$$

It follows that

$$\mathbf{J}\left(E^*\right)=\begin{pmatrix} \frac{\partial G_1}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_1}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_1}{\partial I_h}\left(E^*\right) & \frac{\partial G_1}{\partial I_m}\left(E^*\right) \\ \frac{\partial G_2}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_2}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_2}{\partial I_h}\left(E^*\right) & \frac{\partial G_2}{\partial I_m}\left(E^*\right) \\ \frac{\partial G_3}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_3}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_3}{\partial I_h}\left(E^*\right) & \frac{\partial G_3}{\partial I_m}\left(E^*\right) \\ \frac{\partial G_4}{\partial S_{h_1}}\left(E^*\right) & \frac{\partial G_4}{\partial S_{h_2}}\left(E^*\right) & \frac{\partial G_4}{\partial I_h}\left(E^*\right) & \frac{\partial G_4}{\partial I_m}\left(E^*\right) \end{pmatrix}$$

Hence the variational matrix of the non- linear model system (4.36) is obtained as

$$\mathbf{J}\left(E^*\right)=\begin{pmatrix} -\mu_h-\frac{B_1\beta_{mh}\left(1-\tau_2\right)I_m^*}{N_h} & \theta_2u_1 & 0 & -\frac{B_1\beta_{mh}\left(1-u_2\right)S_{h_1}^*}{N_h} \\ 0 & A & 0 & -\frac{B_2\beta_{mh}\left(1-u_2\right)S_{h_2}^*}{N_h} \\ \frac{B_1\beta_{mh}\left(1-u_2\right)I_m^*}{N_h} & \frac{B_2\beta_{mh}\left(1-u_2\right)I_m^*}{N_h} & -a-\eta_h-\mu_h & B \\ 0 & 0 & C & -\mu_m-u_4-\frac{B_3\beta_{hm}\left(1-u_2\right)I_h^*}{N_h} \end{pmatrix} \quad (4.37)$$

$$A=-\mu_h-\theta_2u_1-\frac{B_2\beta_{mh}\left(1-u_2\right)I_m^*}{N_h},\qquad B=\frac{\beta_{mh}\left(B_1\left(1-u_2\right)S_{h_1}^*+B_2\left(1-u_2\right)S_{h_2}^*\right)}{N_h},$$

$$C=mB_3\beta_{hm}\left(1-u_2\right)-\frac{B_3\beta_{hm}\left(1-u_2\right)I_m^*}{N_h}$$

The following lemma was stated and proved by McCluskey and Driessche, (2004), to demonstrate the local stability of endemic equilibrium point E^*.

Lemma 4.2:

Let $J\left(E^*\right)$ be a 4×4 real matrix. If $tr\left(\mathbf{J}\left(E^*\right)\right)$, $\det\left(\mathbf{J}\left(E^*\right)\right)$ and $\det\left(\mathbf{J}\left(E^*\right)^{[2]}\right)$ are all negative, then all eigenvalues of $\mathbf{J}\left(E^*\right)$ have negative real part.

Using the above Lemma, we will study the stability of the endemic equilibrium.

Theorem 4.3: If $R_c > 1$, the endemic equilibrium E^* of the model (4.34) is locally asymptotically stable in Ω

Proof:

From the Jacobian matrix $\mathbf{J}\left(E^*\right)$ in (4.37), we have

$$tr\left(\mathbf{J}\left(E^*\right)\right) = -\mu_h - \frac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h} - u_1\theta_2 - \mu_h - \frac{B_2\left(1-u_2\right)\beta_{mh}I_m^*}{N_h}$$

$$-a - \eta_h - \mu_h - u_4 - \mu_m - \frac{B_3(1-u_2)\beta_{hm}I_h^*}{N_h} < 0$$

$$\det\left(\mathbf{J}\left(E^*\right)\right) = \begin{vmatrix} -\mu_h - \frac{B_1\beta_{mh}(1-u_2)I_m^*}{N_h} & \theta_2 u_1 & 0 & -\frac{B_1\beta_{mh}\left(1-u_2\right)S_{h_1}^*}{N_h} \\ 0 & A & 0 & -\frac{B_2\beta_{mh}(1-u_2)S_{h_2}^*}{N_h} \\ \frac{B_1\beta_{mh}(1-u_2)I_m^*}{N_h} & \frac{B_2\beta_{mh}(1-u_2)I_m^*}{N_h} & -a-\eta_h-\mu_h & B \\ 0 & 0 & C & -\mu_m - u_4 - \frac{B_3\beta_{hm}(1-u_2)I_h^*}{N_h} \end{vmatrix}$$

Therefore

$$\mathrm{Det}\left(\mathbf{J}\left(E^*\right)\right) = -\frac{1}{N_h^3}\left(\frac{kB_3\beta_{mh}\beta_{hm}t}{\varphi\left(\mu_h + u_1\theta_2\right)\left(\mu_m + u_4\right)^2 R_c^2}\left(N_h\left(\mu_m + u_4\right) + B_3\beta_{hm}\left(1-u_2\right)I_h^*\right)\left(N_h\mu_h + \right.\right.$$
$$B_1\beta_{mh}\left(1-u_2\right)I_m^*\Big)\left(N_h\left(\mu_h + \theta_2 u_1\right)B_2\beta_{mh}\left(1-u_2\right)I_m^*\right) + B_3\beta_{hm}\beta_{mh}\mu_h\left(-1+u_2\right)^2\left(mN_h - I_m^*\right)$$
$$\left.\left.\left(B_2N_h\left(\mu_h + \theta_2 u_1\right)S_{h_2}^* + B_1(N_h\left(\mu_h + \theta_2 u_1\right)S_{h_1}^* + B_2\beta_{mh}\left(1-u_2\right)I_m^*\left(S_{h_1}^* + S_{h_2}^*\right)\right)\right)\right)$$

Therefore $\mathrm{Det}\left(\mathbf{J}\left(E^*\right)\right) < 0$ If $R_c > 1$

Hence trace and determinant of the Jacobian matrix $\mathbf{J}\left(E^*\right)$ are all negative.

The second additive compound matrix is obtained from the following lemma.

Lemma 4.3:

To establish the second additive compound matrix $\left(\mathbf{J}^{[2]}\left(E^*\right)\right)$ of the Jacobian matrix $\mathbf{J}(E^*)$, the following will be considered.

From the Jacobian matrix $\mathbf{J}\left(E^*\right)$, the second additive compound matrix $\left(\mathbf{J}^{[2]}\left(E^*\right)\right)$ is obtained by taking the coefficient of X from:

$$\begin{pmatrix} \det\mathbf{N}[1,2|1,2] & \det\mathbf{N}[1,2|1,3] & \det\mathbf{N}[1,2|1,4] & \det\mathbf{N}[1,2|2,3] & \det\mathbf{N}[1,2|2,4] & \det\mathbf{N}[1,2|3,4] \\ \det\mathbf{N}[1,3|1,2] & \det\mathbf{N}[1,3|1,3] & \det\mathbf{N}[1,3|1,4] & \det\mathbf{N}[1,3|2,3] & \det\mathbf{N}[1,3|2,4] & \det\mathbf{N}[1,3|3,4] \\ \det\mathbf{N}[1,4|1,2] & \det\mathbf{N}[1,4|1,3] & \det\mathbf{N}[1,4|1,4] & \det\mathbf{N}[1,4|2,3] & \det\mathbf{N}[1,4|2,4] & \det\mathbf{N}[1,4|3,4] \\ \det\mathbf{N}[2,3|1,2] & \det\mathbf{N}[2,3|1,3] & \det\mathbf{N}[2,3|1,4] & \det\mathbf{N}[2,3|2,3] & \det\mathbf{N}[2,3|2,4] & \det\mathbf{N}[2,3|3,4] \\ \det\mathbf{N}[2,4|1,2] & \det\mathbf{N}[2,4|1,3] & \det\mathbf{N}[2,4|1,4] & \det\mathbf{N}[2,4|2,3] & \det\mathbf{N}[2,4|2,4] & \det\mathbf{N}[2,4|3,4] \\ \det\mathbf{N}[3,4|1,2] & \det\mathbf{N}[3,4|1,3] & \det\mathbf{N}[3,4|1,4] & \det\mathbf{N}[3,4|2,3] & \det\mathbf{N}[3,4|2,4] & \det\mathbf{N}[3,4|3,4] \end{pmatrix}$$

,

where $N_{ij} = \left[\mathbf{J}\left(E^*\right) + \mathbf{I}X\right]$ and $\mathbf{I}$ is identity matrix. It follows that $\mathbf{N} = \left[\mathbf{J}\left(E^*\right) + \mathbf{I}X\right]$

$$\begin{bmatrix} -\mu_h - \dfrac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h} & u_1\theta_2 & 0 & -\dfrac{B_1\left(1-u_2\right)\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & D & 0 & -\dfrac{B_2\left(1-u_2\right)\beta_{mh}S_{h_2}^*}{N_h} \\ \dfrac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h} & \dfrac{B_2(1-u_2)\beta_{mh}I_m^*}{N_h} & -a-\eta_h-\mu_h & A \\ 0 & 0 & B & C \end{bmatrix} + \begin{bmatrix} X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X \end{bmatrix}$$

This is equivalent to

$$\mathbf{N} = \begin{bmatrix} -\mu_h - \dfrac{B_1\left(1-u_2\right)\beta_{mh}I_m^*}{N_h} + X & u_1\theta_2 & 0 & -\dfrac{B_1\left(1-u_2\right)\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & D+X & 0 & -\dfrac{B_2\left(1-u_2\right)\beta_{mh}S_{h_1}^*}{N_h} \\ \dfrac{B_1\left(1-u_2\right)\beta_{mh}I_m^*}{N_h} & \dfrac{B_2\left(1-u_2\right)\beta_{mh}I_m^*}{N_h} & -a-\eta_h-\mu_h+X & A \\ 0 & 0 & B & C+X \end{bmatrix}$$

$$A=\frac{\beta_{mh}\left(B_1\left(1-u_2\right)S_{h_1}^{*}+B_2\left(1-u_2\right)S_{h_2}^{*}\right)}{N_h}\quad B=mB_3\left(1-u_2\right)\beta_{hm}-\frac{B_3\left(1-u_2\right)\beta_{hm}I_m^{*}}{N_h}$$

$$C=-u_4-\mu_m-\frac{B_3\left(1-u_2\right)\beta_{hm}I_h^{*}}{N_h}\quad D=-u_1\theta_2-\mu_h-\frac{B_2\left(1-u_2\right)\beta_{mh}I_m^{*}}{N_h} \tag{4.38}$$

Then $\det \mathbf{N}[1,2\,|\,1,2]=\begin{vmatrix}-\mu_h-\dfrac{B_1\left(1-u_2\right)\beta_{mh}I_m^{*}}{N_h}+X & u_1\theta_2\\ 0 & D+X\end{vmatrix}$ or

$$\det \mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h-\frac{B_1\left(1-u_2\right)\beta_{mh}I_m^{*}}{N_h}+X\right)\left(D+X\right)$$

But from (4.38) $D=-u_1\theta_2-\mu_h-\dfrac{B_2\left(1-u_2\right)\beta_{mh}I_m^{*}}{N_h}$ then

$$\det \mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h+\frac{B_1(-1+u_2)\beta_{mh}I_m^{*}}{N_h}+X\right)\left(-(1-u_1)\theta_2-\mu_h-\frac{B_2(1-u_2)\beta_{mh}I_m^{*}}{N_h}+X\right)\text{or}$$

$$\det \mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h-\frac{B_1(1-u_2)\beta_{mh}I_m^{*}}{N_h}\right)\left(-(1-u_1)\theta_2-\mu_h-\frac{B_2(1-u_2)\beta_{mh}I_m^{*}}{N_h}\right)+$$

$$\left(-\mu_h-\frac{B_1(1-u_2)\beta_{mh}I_m^{*}}{N_h}\right)X+\left(-(1-u_1)\theta_2-\mu_h-\frac{B_2(1-u_2)\beta_{mh}I_m^{*}}{N_h}\right)X+X^2$$

Take coefficient of X as

$$\det \mathbf{N}[1,2\,|\,1,2]=\left(-\mu_h-\frac{B_1(1-u_2)\beta_{mh}I_m^{*}}{N_h}\right)+\left(-(1-u_1)\theta_2-\mu_h-\frac{B_2(1-u_2)\beta_{mh}I_m^{*}}{N_h}\right)$$

Therefore $\det \mathbf{N}[1,2\,|\,1,2]=-\mu_h-\dfrac{B_1(1-u_2)\beta_{mh}I_m^{*}}{N_h}-(1-u_1)\theta_2-\mu_h-\dfrac{B_2(1-u_2)\beta_{mh}I_m^{*}}{N_h}$

Others will be calculated in the same way to obtain the following matrix:

$$\mathbf{J}^{[2]}(E^*)=\begin{pmatrix} a_{11} & 0 & -\frac{B_2(1-u_2)\beta_{mh}S_{h_2}^*}{N_h} & 0 & \frac{B_1(1-u_2)\beta_{mh}S_{h_1}^*}{N_h} & 0 \\ \frac{B_2(1-u_2)\beta_{mh}I_m^*}{N_h} & a_{22} & a_{23} & u_1\theta_2 & 0 & \frac{B_1(1-u_2)\beta_{mh}S_{h_1}^*}{N_h} \\ 0 & a_{32} & a_{33} & 0 & u_1\theta_2 & 0 \\ -\frac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h} & 0 & 0 & a_{44} & a_{45} & \frac{B_2(1-u_2)\beta_{mh}S_{h_2}^*}{N_h} \\ 0 & 0 & 0 & a_{54} & a_{55} & 0 \\ 0 & 0 & \frac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h} & 0 & \frac{B_2(1-u_2)\beta_{mh}I_m^*}{N_h} & a_{66} \end{pmatrix}$$

$$a_{11}=-2\mu_h-\frac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h}-u_1\theta_2-\frac{B_2(1-u_2)\beta_{mh}I_m^*}{N_h},$$

$$a_{22}=-\frac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h}-a-\eta_h-2\mu_h$$

$$a_{23}=\frac{\beta_{mh}(B_1(1-u_2)S_{h_1}^*+B_2(1-u_2)S_{h_2}^*)}{N_h}\ ,\quad a_{32}=mB_3(1-u_2)\beta_{hm}-\frac{B_3(1-u_2)\beta_{hm}I_m^*}{N_h}$$

$$a_{33}=-\mu_h-\frac{B_1(1-u_2)\beta_{mh}I_m^*}{N_h}-u_4-\mu_m-\frac{B_3(1-u_2)\beta_{hm}I_h^*}{N_h},$$

$$a_{44}=-2\mu_h-u_1\theta_2-\frac{B_2(1-u_2)\beta_{mh}I_m^*}{N_h}-a-\eta_h$$

$$a_{45}=\frac{\beta_{mh}(B_1(1-u_2)S_{h_1}^*+B_2(1-u_2)S_{h_2}^*)}{N_h},a_{54}=mB_3(1-u_2)\beta_{hm}-\frac{B_3(1-u_2)\beta_{hm}I_m^*}{N_h}$$

$$a_{55}=-\mu_h-\theta_2u_1-\frac{B_2\beta_{mh}(1-u_2)I_m^*}{N_h}-\mu_m-u_4-\frac{B_3\beta_{hm}(1-u_2)I_h^*}{N_h},$$

$$a_{66}=-a-\eta_h-\mu_h-\mu_m-u_4-\frac{B_3\beta_{hm}(1-u_2)I_h^*}{N_h}$$

Thus

$$\det\left(\mathbf{J}_M^{[2]}\left(E^*\right)\right)=-\frac{1}{N_h^6}\Big(\left(1-u_2\right)^2\alpha_5R_c^2\left(mN_h-I_m^*\right)\Big(\left(1-u_2\right)^2B_3\beta_{hm}\beta_{mh}\left(mN_h-I_m^*\right)f+\Big(N_h\left(a+\eta_h+2\mu_h\right)$$
$$+\left(1-u_2\right)B_1\beta_{mh}I_m^*\Big)\Big(\Big(N_h\left(a+\eta_h+\mu_h+\mu_m+u_4\right)+\left(1-u_2\right)B_3\beta_{hm}I_h^*\Big)\left(\sqrt{\alpha}R_c+1\right)\left(\sqrt{\alpha}R_c-1\right)\Big(w+\left(1-u_2\right)$$

$$B_1\beta_{mh}I_m^*\Big(B_1N_h\left(\mu_h+\mu_m+u_4\right)\left(\sqrt{\alpha_2}R_c+1\right)\left(\sqrt{\alpha_2}R_c-1\right)S_{h_1}^*+\alpha_3R_c^2\left(1-u_2\right)B_1^2I_m^*S_{h_1}^*+u_1B_2N_h\theta_2S_{h_2}^*\Big)\Big)\Big)\Big)\Big)+$$

$$\left(N_h\left(u_1\theta_2+\mu_h+\mu_m+u_4\right)+\left(1-u_2\right)\left(B_3\beta_{hm}I_h^*+B_2\beta_{mh}I_m^*\right)\right)\Big(j+N_h\left(a+\eta_h+\mu_h+\mu_m+u_4\right)\left(\sqrt{\alpha_4}R_c+1\right)$$

$$\left(\sqrt{\alpha_4}R_c-1\right)\Big(\alpha_5R_c^2\left(1-u_2\right)^3u_1B_1B_2N_h\beta_{mh}\theta_2I_m^*\left(mN_h-I_m^*\right)S_{h_2}^*+\Big(N_h\left(a+\eta_h+u_1\theta_2+2\mu_h\right)+\left(1-u_2\right)$$

$$B_2\beta_{mh}I_m^*\Big)\Big(\Big(N_h\left(a+\eta_h+2\mu_h\right)+\left(1-u_2\right)B_1\beta_{mh}I_m^*\Big)y-\left(1-u_2\right)^2B_3\beta_{hm}\beta_{mh}\left(mN_h-I_m^*\right)\Big(\alpha_6R_c^2\left(1-u_2\right)$$

$$B_2^2I_m^*S_{h_2}^*+\Big(N_h(u_1\theta_2+2\mu_h)+\left(1-u_2\right)\left(B_1+B_2\right)\beta_{mh}I_m^*\Big)\Big(B_1S_{h_1}^*+B_2S_{h_2}^*\Big)\Big)\Big)\Big)\Big)\Big)\Big)$$

where $\alpha=\dfrac{\left(1-u_2\right)B_2I_m^*\left(\mu_h+\eta_h+a\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2r}{\left(N_h\left(a+\eta_h+\mu_h+\mu_m+u_4\right)+\left(1-u_2\right)B_3\beta_{hm}I_h^*\right)B_3\beta_{hm}kt}$

$$\alpha_2=\frac{B_1\left(1-u_2\right)I_h^*\left(\mu_h+\eta_h+a\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2}{B_1N_h\left(\mu_h+\mu_m+u_4\right)\beta_{mh}kt}\quad\alpha_3=\frac{\left(\mu_h+\eta_h+a\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2}{B_3\beta_{hm}kt}$$

$$\alpha_4=\frac{\left(1-u_2\right)I_h^*\left(\mu_h+\eta_h+a\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2}{N_h\left(a+\eta_h+\mu_h+\mu_m+u_4\right)\beta_{mh}kt}\quad\alpha_5=\frac{\left(\mu_h+\eta_h+a\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2}{kt}$$

$$\alpha_6=\frac{\left(\mu_h+\eta_h+a\right)\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2}{B_3\beta_{hm}kt}\quad R_c^2=\frac{-kB_3\beta_{mh}\beta_{hm}t}{\varphi\left(\mu_h+u_1\theta_2\right)\left(\mu_m+u_4\right)^2\left(\mu_h+\eta_h+a\right)}$$

$$f=(B_1\Big(N_h\left(a+\eta_h+\mu_h+\mu_m+u_4\right)+\left(1-u_2\right)B_3\beta_{hm}I_h^*\Big)S_{h_1}^*+\left(1-u_2\right)B_1^2\beta_{mh}I_m^*S_{h_1}^*+$$

$$B_2\Big(N_h\left(a+\eta_h+\mu_h+\mu_m+u_4\right)+\left(1-u_2\right)\left(B_3\beta_{hm}I_h^*+B_2\beta_{mh}I_m^*\right)\Big)S_{h_2}^*\Big)\Big(B_2N_h$$

$$\left(u_1\theta_2+2\mu_h\right)S_{h_2}^*+B_1\Big(N_h\left(u_1\theta_2+2\mu_h\right)S_{h_1}^*+\left(1-u_2\right)B_2\beta_{mh}I_m^*\left(S_{h_1}^*+S_{h_2}^*\right)\Big)\Big)$$

$$r=\Big(N_h\left(u_1\theta_2+2\mu_h\right)+\left(1-u_2\right)\left(B_1+B_2\right)\beta_2I_m^*\Big)\Big(B_2\Big(N_h\left(\mu_h+\mu_m+u_4\right)$$

$$+\left(1-u_2\right)B_3\beta_{hm}I_h^*\Big)+B_1\left(u_1N_h\theta_2+\left(1-u_2\right)B_2\beta_{mh}I_m^*\right)\Big)S_{h_2}^*$$

$$w=\Big(N_h\left(u_1\theta_2+2\mu_h\right)+\left(1-u_2\right)\left(B_1+B_2\right)\beta_{mh}I_m^*\Big)\Big(N_h\left(\mu_h+\mu_m+u_4\right)+$$

$$\left(1-u_2\right)\left(B_3\beta_{hm}I_h^*+B_1\beta_{mh}I_m^*\right)\Big)\Big(B_1S_{h_1}^*+B_2S_{h_2}^*\Big)$$

$$j=\left(1-u_2\right)^3B_1B_3\beta_{hm}\beta_{mh}^2\left(mN_h-I_m^*\right)I_m^*\Big(N_h\left(u_1\theta_2+2\mu_h\right)+\left(1-u_2\right)\left(B_1+B_2\right)\beta_{mh}I_m^*\Big)$$

$$\Big(B_1\Big(N_h\left(a+\eta_h+u_1\theta_2+2\mu_h\right)+\left(1-u_2\right)B_2\beta_{mh}I_m^*\Big)S_{h_1}^*+u_1B_2N_h\theta_2S_{h_2}^*\Big)$$

$$y=\left(N_h\left(u_1\theta_2+2\mu_h\right)+\left(1-u_2\right)\left(B_1+B_2\right)\beta_{mh}I_m^*\right)\left(N_h\left(\mu_h+\mu_m+u_4\right)+\left(1-u_2\right)\left(B_3\beta_{hm}I_h^*+B_1\beta_{mh}I_m^*\right)\right)$$

Hence $\det\left(\mathbf{J}_M^{[2]}\left(E^*\right)\right)<0$, if $R_c>1$.

Thus, from the Lemma 4.2, the endemic equilibrium E^* of the model system (4.34) is locally asymptotically stable in Ω.

4.8.3: Global Stability of Endemic Equilibrium Point (EEP)

Theorem 4.4:

If $R_c>1$ the endemic equilibrium E^* of the model system (4.1) is globally asymptotically stable.

Proof:

To establish the global stability of endemic equilibrium E^* we construct the following positive Lyapunov function V as follows:

$$V\left(S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*\,\&\,\mathrm{I}_m^*\right)=\left(S_{h_1}-S_{h_1}^*\ln S_{h_1}\right)+\left(S_{h_2}-S_{h_2}^*\ln S_{h_2}\right)+\left(I_h-I_h^*\ln I_h\right)+$$
$$\left(T_h-T_h^*\ln T_h\right)+\left(R_h-R_h^*\ln R_h\right)+\left(A_m-A_m^*\ln A_m\right)+\left(S_m-S_m^*\ln S_m\right)+\left(I_m-I_m^*\ln I_m\right)\quad(4.39)$$

Direct calculation of the derivative of V along the solutions of (4.39) gives

$$\frac{dV}{dt}\left(S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*\,\&\,\mathrm{I}_m^*\right)=\left(1-\frac{S_{h_1}^*}{S_{h_1}}\right)\frac{dS_{h_1}}{dt}+\left(1-\frac{S_{h_2}^*}{S_{h_2}}\right)\frac{dS_{h_2}}{dt}+\left(1-\frac{I_h^*}{I_h}\right)\frac{dI_h}{I_h}+$$
$$\left(1-\frac{T_h^*}{T_h}\right)\frac{dT_h}{dt}+\left(1-\frac{R_h^*}{R_h}\right)\frac{dR_h}{dt}+\left(1-\frac{A_m^*}{A_m}\right)\frac{dA_m}{dt}+\left(1-\frac{S_m^*}{S_m}\right)\frac{dS_m}{S_m}+\left(1-\frac{I_m^*}{I_m}\right)\frac{dI_m}{dt}$$

Consequently

$$\frac{dV}{dt}=\left(\frac{S_{h_1}-S_{h_1}^*}{S_{h_1}}\right)\frac{dS_{h_1}}{dt}+\left(\frac{S_{h_2}-S_{h_2}^*}{S_{h_2}}\right)\frac{dS_{h_2}}{dt}+\left(\frac{I_h-I_h^*}{I_h}\right)\frac{dI_h}{dt}+\left(\frac{T_h-T_h^*}{T_h}\right)\frac{dT_h}{dt}+$$

$$\left(\frac{R_h-R_h^*}{R_h}\right)\frac{dR_h}{dt}+\left(\frac{A_m-A_m^*}{A_m}\right)\frac{dA_m}{dt}+\left(\frac{S_m-S_m^*}{S_m}\right)\frac{dS_m}{dt}+\left(\frac{I_m-I_m^*}{I_m}\right)\frac{dI_m}{dt}$$

Substituting $S_{h_1} = S_{h_1} - S_{h_1}^*$, $S_{h_2} = S_{h_2} - S_{h_2}^*$, $I_h = I_h - I_h^*$, $T_h = T_h - T_h^*$, $R_h = R_h - R_h^*$,

$A_m = A_m - A_m^*$, $S_m = S_m - S_m^*$ and $I_m = I_m - I_m^*$ into (4.1) gives

$$\frac{dS_{h_1}}{dt} = (1-\pi)\mu_h N_h - (1-u_2) B_1 \beta_{mh} \frac{(I_m - I_m^*)}{N_h}(S_{h_1} - S_{h_1}^*) - \mu_h (S_{h_1} - S_{h_1}^*) + \theta_1 (R_h - R_h^*) + u_1\theta_2 (S_{h_2} - S_{h_2}^*)$$

$$\frac{dS_{h_2}}{dt} = \pi\mu_h N_h - (1-u_2) B_2 \beta_{mh} \frac{(I_m - I_m^*)}{N_h}(S_{h_2} - S_{h_2}^*) - (\mu_h + u_1\theta_2)(S_{h_2} - S_{h_2}^*)$$

$$\frac{dI_h}{dt} = \left((1-u_2) B_1 (S_{h_1} - S_{h_1}^*) + (1-u_2) B_2 (S_{h_2} - S_{h_2}^*)\right)\beta_{mh} \frac{(I_m - I_m^*)}{N_h} - (\mu_h + \eta_h + a)(I_h - I_h^*)$$

$$\frac{dT_h}{dt} = \eta_h (I_h - I_h^*) - (\mu_h + \delta_h)(T_h - T_h^*)$$

$$\frac{dR_h}{dt} = \delta_h (T_h - T_h^*) - (\mu_h + \theta_1)(R_h - R_h^*)$$

$$\frac{dA_m}{dt} = \left(\varphi - \varphi \frac{(A_m - A_m^*)}{kN_h}\right)\left((S_m - S_m^*) + (I_m - I_m^*)\right) - \left(\mu_A + (1-u_5)\eta_A + u_3\right)(A_m - A_m^*)$$

$$\frac{dS_m}{dt} = (1-u_5)\eta_A (A_m - A_m^*) - \left((1-u_2) B_3 \beta_{hm} \frac{(I_h - I_h^*)}{N_h}\right)(S_m - S_m^*) - (u_4 + \mu_m)(S_m - S_m^*)$$

$$\frac{dI_m}{dt} = (1-u_2) B_3 \beta_{hm} \frac{(I_h - I_h^*)}{N_h}(S_m - S_m^*) - (\mu_m + u_4)(I_m - I_m^*)$$

It is follows that

$$\frac{dV}{dt} = j_1 + j_2 + j_3 + j_4 + j_5 + j_6 + j_7 + j_8 \tag{4.40}$$

where,

$$j_1 = \left(\frac{S_{h_1} - S_{h_1}^*}{S_{h_1}}\right)\left\{(1-\pi)\mu_h N_h - (1-u_2) B_1 \beta_{mh} \frac{(I_m - I_m^*)}{N_h}(S_{h_1} - S_{h_1}^*) - \mu_h (S_{h_1} - S_{h_1}^*) + \theta_1 (R_h - R_h^*) + u_1\theta_2 (S_{h_2} - S_{h_2}^*)\right\} \tag{4.41}$$

$$j_2=\left(\frac{S_{h_2}-S_{h_2}^*}{S_{h_2}}\right)\left\{\pi\mu_h N_h-(1-u_2)B_2\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}\left(S_{h_2}-S_{h_2}^*\right)-\left(\mu_h+u_1\theta_2\right)\left(S_{h_2}-S_{h_2}^*\right)\right\} \quad (4.42)$$

$$j_3=\left(\frac{I_h-I_h^*}{I_h}\right)\left\{\left((1-u_2)B_1\left(S_{h_1}-S_{h_1}^*\right)+(1-u_2)B_2\left(S_{h_2}-S_{h_2}^*\right)\right)\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}-\left(\mu_h+\eta_h+a\right)\left(I_h-I_h^*\right)\right\} (4.43)$$

$$j_4=\left(\frac{T_h-T_h^*}{T_h}\right)\left\{\eta_h\left(I_h-I_h^*\right)-\left(\mu_h+\delta_h\right)\left(T_h-T_h^*\right)\right\} \quad (4.44)$$

$$j_5=\left(\frac{R_h-R_h^*}{R_h}\right)\left\{\delta_h\left(T_h-T_h^*\right)-\left(\mu_h+\theta_1\right)\left(R_h-R_h^*\right)\right\} \quad (4.45)$$

$$j_6=\left(\frac{A_m-A_m^*}{A_m}\right)\left\{\left(\varphi-\varphi\frac{\left(A_m-A_m^*\right)}{kN_h}\right)\left(\left(S_m-S_m^*\right)+\left(I_m-I_m^*\right)\right)-\left(\mu_A+(1-u_5)\eta_A+u_3\right)\left(A_m-A_m^*\right)\right\} (4.46)$$

$$j_7=\left(\frac{S_m-S_m^*}{S_m}\right)\left\{(1-u_5)\eta_A\left(A_m-A_m^*\right)-\left((1-u_2)B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}\right)\left(S_m-S_m^*\right)-\left(u_4+\mu_m\right)\left(S_m-S_m^*\right)\right\} (4.47)$$

$$j_8=\left(\frac{I_m-I_m^*}{I_m}\right)\left\{(1-u_2)B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}\left(S_m-S_m^*\right)-\left(\mu_m+u_4\right)\left(I_m-I_m^*\right)\right\} \quad (4.48)$$

We simplify (4.41) to (4.48)

From (4.41) we have

$$j_1=\left(\frac{S_{h_1}-S_{h_1}^*}{S_{h_1}}\right)\left\{(1-\pi)\mu_h N_h-(1-u_2)B_1\beta_{mh}\frac{\left(I_m-I_m^*\right)}{N_h}\left(S_{h_1}-S_{h_1}^*\right)-\mu_h\left(S_{h_1}-S_{h_1}^*\right)+\theta_1\left(R_h-R_h^*\right)+u_1\theta_2\left(S_{h_2}-S_{h_2}^*\right)\right\}$$

$$j_1=\left(\frac{S_{h_1}-S_{h_1}^*}{S_{h_1}}\right)\left\{\begin{array}{l}\mu_h N_h-\pi\mu_h N_h-\left(\left(\frac{B_1\beta_{mh}I_m}{N_h}-\frac{B_1\beta_{mh}I_m^*}{N_h}\right)\left(S_{h_1}-S_{h_1}^*\right)-\left(\frac{u_2B_1\beta_{mh}I_m}{N_h}-\frac{u_2B_1\beta_{mh}I_m^*}{N_h}\right)\left(S_{h_1}-S_{h_1}^*\right)\right)\\ -\mu_h\left(S_{h_1}-S_{h_1}^*\right)+\left(\theta_1R_h-\theta_1R_h^*\right)+u_1\theta_2S_{h_2}-u_1\theta_2S_{h_2}^*\end{array}\right\}$$

$$j_1 = \left\{ \mu_h N_h \left(1 - \frac{S_{h_1}^*}{S_{h_1}}\right) - \pi\mu_h N_h \left(1 - \frac{S_{h_1}^*}{S_{h_1}}\right) - \left(\left(\frac{B_1\beta_{mh} I_m}{N_h} - \frac{B_1\beta_{mh} I_m^*}{N_h} \right) \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \left(\frac{u_2 B_1\beta_{mh} I_m}{N_h} - \frac{u_2 B_1\beta_{mh} I_m^*}{N_h} \right) \right.\right.$$

$$\left.\left. \times \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} \right) - \mu_h \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} + \left(\theta_1 R_h - \theta_1 R_h^*\right)\left(1 - \frac{S_{h_1}^*}{S_{h_1}}\right) + u_1\theta_2 S_{h_2}\left(1 - \frac{S_{h_1}^*}{S_{h_1}}\right) - u_1\theta_2 S_{h_2}^*\left(1 - \frac{S_{h_1}^*}{S_{h_1}}\right) \right\}$$

$$j_1 = \mu_h N_h + \pi\mu_h N_h \frac{S_{h_1}^*}{S_{h_1}} + \frac{B_1\beta_{mh} I_m^*}{N_h} \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} + \frac{u_2 B_1\beta_{mh} I_m}{N_h} \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} + \theta_1 R_h + \theta_1 R_h^* \frac{S_{h_1}^*}{S_{h_1}} +$$

$$u_1\theta_2 S_{h_2} + u_1\theta_2 S_{h_2}^* \frac{S_{h_1}^*}{S_{h_1}} - \mu_h N_h \frac{S_{h_1}^*}{S_{h_1}} - \pi\mu_h N_h - \frac{B_1\beta_{mh} I_m}{N_h} \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \frac{u_2 B_1\beta_{mh} I_m^*}{N_h} \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}}$$

$$-\mu_h \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \theta_1 R_h^* - \theta_1 R_h \frac{S_{h_1}^*}{S_{h_1}} - u_1\theta_2 S_{h_2} \frac{S_{h_1}^*}{S_{h_1}} - u_1\theta_2 S_{h_2}^*$$

From (4.42) we have

$$j_2 = \left(\frac{S_{h_2} - S_{h_2}^*}{S_{h_2}} \right) \left\{ \pi\mu_h N_h - (1 - u_2) B_2\beta_{mh} \frac{\left(I_m - I_m^*\right)}{N_h} \left(S_{h_2} - S_{h_2}^*\right) - \left(\mu_h + u_1\theta_2\right)\left(S_{h_2} - S_{h_2}^*\right) \right\}.$$

It follows that

$$j_2 = \pi\mu_h N_h \left(1 - \frac{S_{h_2}^*}{S_{h_2}}\right)\left(-B_2\beta_{mh} \frac{I_m}{N_h} + u_2 B_2\beta_{mh} \frac{I_m}{N_h} + B_2\beta_{mh} \frac{I_m^*}{N_h} - u_2 B_2\beta_{mh} \frac{I_m^*}{N_h} \right) \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - \left(\mu_h + u_1\theta_2\right) \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}}.$$

Consequently

$$j_2 = \pi\mu_h N_h + u_2 B_2\beta_{mh} \frac{I_m}{N_h} \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} + B_2\beta_{mh} \frac{I_m^*}{N_h} \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - \pi\mu_h N_h \frac{S_{h_2}^*}{S_{h_2}} -$$

$$B_2\beta_{mh} \frac{I_m}{N_h} \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - u_2 B_2\beta_{mh} \frac{I_m^*}{N_h} \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - \left(\mu_h + u_1\theta_2\right) \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}}$$

From (4.43) we have

$$j_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left((1-u_2)B_1\left(S_{h_1} - S_{h_1}^*\right) + (1-u_2)B_2\left(S_{h_2} - S_{h_2}^*\right)\right)\beta_{mh}\frac{\left(I_m - I_m^*\right)}{N_h} - (\mu_h + \eta_h + a)\left(I_h - I_h^*\right)\right\}$$

or

$$j_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left(\left(B_1 S_{h_1} - u_2 B_1 S_{h_1} - B_1 S_{h_1}^* + u_2 B_1 S_{h_1}^*\right)\left(\frac{\beta_{mh} I_m}{N_h} - \frac{\beta_{mh} I_m^*}{N_h}\right) + \left(B_2 S_{h_2} - u_2 B_2 S_{h_2}\right.\right.\right.$$

$$\left.\left.\left. - B_2 S_{h_2}^* + u_2 B_2 S_{h_2}^*\right)\left(\frac{\beta_{mh} I_m}{N_h} - \frac{\beta_{mh} I_m^*}{N_h}\right)\right) - (\mu_h + \eta_h + a)\left(I_h - I_h^*\right)\right\}$$

$$j_3 = \left(\frac{I_h - I_h^*}{I_h}\right)\left\{\left(B_1 S_{h_1}\frac{\beta_{mh} I_m}{N_h} - u_2 B_1 S_{h_1}\frac{\beta_{mh} I_m}{N_h} - B_1 S_{h_1}^*\frac{\beta_{mh} I_m}{N_h} + u_2 B_1 S_{h_1}^*\frac{\beta_{mh} I_m}{N_h} - \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1} + \right.\right.$$

$$\frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1} + \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1}^* - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1}^* + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} - \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2} -$$

$$\frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* + \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2}^* - \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2} + \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2} + \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}^* -$$

$$\left.\left.\frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2}^*\right) - (\mu_h + \eta_h + a)\left(I_h - I_h^*\right)\right\}$$

then

$$j_3 = \left\{\left(B_1 S_{h_1}\frac{\beta_{mh} I_m}{N_h}\left(1 - \frac{I_h^*}{I_h}\right) - u_2 B_1 S_{h_1}\frac{\beta_{mh} I_m}{N_h}\left(1 - \frac{I_h^*}{I_h}\right) - B_1 S_{h_1}^*\frac{\beta_{mh} I_m}{N_h}\left(1 - \frac{I_h^*}{I_h}\right) + u_2 B_1 S_{h_1}^*\frac{\beta_{mh} I_m}{N_h}\left(1 - \frac{I_h^*}{I_h}\right) - \frac{\beta_{mh} I_m^*}{N_h} \times\right.\right.$$

$$B_1 S_{h_1}\left(1 - \frac{I_h^*}{I_h}\right) + \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1}\left(1 - \frac{I_h^*}{I_h}\right) + \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1}^*\left(1 - \frac{I_h^*}{I_h}\right) - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1}^*\left(1 - \frac{I_h^*}{I_h}\right) + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}\left(1 - \frac{I_h^*}{I_h}\right)$$

$$-\frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2}\left(1 - \frac{I_h^*}{I_h}\right) - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^*\left(1 - \frac{I_h^*}{I_h}\right) + \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2}^*\left(1 - \frac{I_h^*}{I_h}\right) - \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}\left(1 - \frac{I_h^*}{I_h}\right) +$$

$$\left.\left.\frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2}\left(1 - \frac{I_h^*}{I_h}\right) + \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}^*\left(1 - \frac{I_h^*}{I_h}\right) - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2}^*\left(1 - \frac{I_h^*}{I_h}\right)\right) - (\mu_h + \eta_h + a)\frac{\left(I_h - I_h^*\right)^2}{I_h}\right\}$$

or

$$j_3 = B_1 S_{h_1} \frac{\beta_{mh} I_m}{N_h} + u_2 B_1 S_{h_1} \frac{\beta_{mh} I_m}{N_h} \frac{I_h^*}{I_h} + B_1 S_{h_1}^* \frac{\beta_{mh} I_m}{N_h} \frac{I_h^*}{I_h} + u_2 B_1 S_{h_1}^* \frac{\beta_{mh} I_m}{N_h} + \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1} \frac{I_h^*}{I_h} + \frac{\beta_{mh} I_m^*}{N_h} \times$$

$$u_2 B_1 S_{h_1} + \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1}^* + \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1}^* \frac{I_h^*}{I_h} + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} + \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2} \frac{I_h^*}{I_h} + \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* \frac{I_h^*}{I_h}$$

$$+ \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2}^* + \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2} \frac{I_h^*}{I_h} + \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2} + \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}^* + \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2}^* \frac{I_h^*}{I_h}$$

$$- B_1 S_{h_1} \frac{\beta_{mh} I_m}{N_h} \frac{I_h^*}{I_h} - u_2 B_1 S_{h_1} \frac{\beta_{mh} I_m}{N_h} - B_1 S_{h_1}^* \frac{\beta_{mh} I_m}{N_h} - u_2 B_1 S_{h_1}^* \frac{\beta_{mh} I_m}{N_h} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1}$$

$$- \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1}^* \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1}^* - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m}{N_h} \times$$

$$u_2 B_2 S_{h_2} - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* - \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2}^* \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2} - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2} \frac{I_h^*}{I_h} -$$

$$\frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}^* \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2}^* - \left(\mu_h + \eta_h + a\right) \frac{\left(I_h - I_h^*\right)^2}{I_h}.$$

From (4.44) we have

$$j_4 = \left(\frac{T_h - T_h^*}{T_h}\right) \left\{ \eta_h \left(I_h - I_h^*\right) - \left(\mu_h + \delta_h\right)\left(T_h - T_h^*\right) \right\}.$$

It follows that

$$j_4 = \left\{ \left(\eta_h I_h - \eta_h I_h^*\right)\left(1 - \frac{T_h^*}{T_h}\right) - \left(\mu_h + \delta_h\right) \frac{\left(T_h - T_h^*\right)^2}{T_h} \right\}.$$

Consequently

$$j_4 = \eta_h I_h + \eta_h I_h^* \frac{T_h^*}{T_h} - \eta_h I_h \frac{T_h^*}{T_h} - \eta_h I_h^* - \left(\mu_h + \delta_h\right) \frac{\left(T_h - T_h^*\right)^2}{T_h}.$$

From (4.45) we have

$$j_5=\left(\frac{R_h-R_h^*}{R_h}\right)\left\{\delta_h\left(T_h-T_h^*\right)-\left(\mu_h+\theta_1\right)\left(R_h-R_h^*\right)\right\} \text{ or}$$

$$j_5=\left\{\left(\delta_h T_h-\delta_h T_h^*\right)\left(1-\frac{R_h^*}{R_h}\right)-\left(\mu_h+\theta_1\right)\frac{\left(R_h-R_h^*\right)^2}{R_h}\right\}$$

Thus

$$j_5=\delta_h T_h+\delta_h T_h^*\frac{R_h^*}{R_h}-\delta_h T_h^*-\delta_h T_h\frac{R_h^*}{R_h}-\left(\mu_h+\theta_1\right)\frac{\left(R_h-R_h^*\right)^2}{R_h}.$$

From (4.46) we have

$$j_6=\left(\frac{A_m-A_m^*}{A_m}\right)\left\{\left(\varphi-\varphi\frac{\left(A_m-A_m^*\right)}{kN_h}\right)\left(\left(S_m-S_m^*\right)+\left(I_m-I_m^*\right)\right)-\left(\mu_A+(1-u_5)\eta_A+u_3\right)\left(A_m-A_m^*\right)\right\} \text{or}$$

$$j_6=\left\{\left(\left(\left(\varphi S_m\left(1-\frac{A_m^*}{A_m}\right)-\varphi S_m^*\left(1-\frac{A_m^*}{A_m}\right)-\varphi S_m\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}+\varphi S_m^*\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}\right)+\left(\varphi I_m\left(1-\frac{A_m^*}{A_m}\right)-\right.\right.\right.\right.$$

$$\left.\left.\left.\varphi I_m^*\left(1-\frac{A_m^*}{A_m}\right)-\varphi I_m\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}+\varphi I_m^*\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}\right)\right)\left(-\mu_A-\eta_A+u_5\eta_A-u_3\right)\frac{\left(A_m-A_m^*\right)^2}{A_m}\right\}$$

then

$$j_6=\varphi S_m+\varphi S_m^*\frac{A_m^*}{A_m}+\varphi S_m^*\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}+\varphi I_m+\varphi I_m^*\frac{A_m^*}{A_m}+\varphi I_m^*\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}+u_5\eta_A\frac{\left(A_m-A_m^*\right)^2}{A_m}$$

$$-\varphi S_m\frac{A_m^*}{A_m}-\varphi S_m^*-\varphi S_m\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}-\varphi I_m\frac{A_m^*}{A_m}-\varphi I_m^*-\varphi I_m\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}-\left(\mu_A+\eta_A+u_3\right)\frac{\left(A_m-A_m^*\right)^2}{A_m}$$

From (4.47) we have

$$j_7=\left(\frac{S_m-S_m^*}{S_m}\right)\left\{(1-u_5)\eta_A\left(A_m-A_m^*\right)-\left(\left(1-u_2\right)B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}\right)\left(S_m-S_m^*\right)-\left(u_4+\mu_m\right)\left(S_m-S_m^*\right)\right\}$$

or

$$j_7=\eta_A A_m\left(1-\frac{S_m^*}{S_m}\right)-u_5\eta_A A_m\left(1-\frac{S_m^*}{S_m}\right)-A_m^*\eta_A\left(1-\frac{S_m^*}{S_m}\right)+A_m^*u_5\eta_A\left(1-\frac{S_m^*}{S_m}\right)-\left(B_3\beta_{hm}\frac{I_h}{N_h}-\right.$$
$$\left.u_2B_3\beta_{hm}\frac{I_h}{N_h}-\frac{I_h^*}{N_h}B_3\beta_{hm}+\frac{I_h^*}{N_h}u_2B_3\beta_{hm}\right)\frac{\left(S_m-S_m^*\right)^2}{S_m}-\left(u_4+\mu_m\right)\frac{\left(S_m-S_m^*\right)^2}{S_m}$$

It follows that

$$j_7=\eta_A A_m+u_2B_3\beta_{hm}\frac{I_h}{N_h}\frac{\left(S_m-S_m^*\right)^2}{S_m}+B_3\beta_{hm}\frac{I_h^*}{N_h}\frac{\left(S_m-S_m^*\right)^2}{S_m}+\eta_Au_5A_m\frac{S_m^*}{S_m}+A_m^*\eta_A\frac{S_m^*}{S_m}+A_m^*\eta_Au_5-\eta_AA_m\frac{S_m^*}{S_m}$$
$$-\eta_Au_5A_m-A_m^*\eta_A-A_m^*\eta_Au_5\frac{S_m^*}{S_m}-B_3\beta_{hm}\frac{I_h}{N_h}\frac{\left(S_m-S_m^*\right)^2}{S_m}-u_2B_3\beta_{hm}\frac{I_h^*}{N_h}\frac{\left(S_m-S_m^*\right)^2}{S_m}-\left(u_4+\mu_m\right)\frac{\left(S_m-S_m^*\right)^2}{S_m}.$$

From (4.48) we have

$$j_8=\left(\frac{I_m-I_m^*}{I_m}\right)\left\{\left(1-u_2\right)B_3\beta_{hm}\frac{\left(I_h-I_h^*\right)}{N_h}\left(S_m-S_m^*\right)-\left(\mu_m+u_4\right)\left(I_m-I_m^*\right)\right\}$$

or

$$j_8=\left(\frac{I_m-I_m^*}{I_m}\right)\left\{\left(B_3\beta_{hm}-u_2B_3\beta_{hm}\right)\left(\frac{I_h}{N_h}-\frac{I_h^*}{N_h}\right)\left(S_m-S_m^*\right)-\left(\mu_m+u_4\right)\left(I_m-I_m^*\right)\right\}.$$

Then

$$j_8=\left(\frac{I_m-I_m^*}{I_m}\right)\left\{\left(B_3\beta_{hm}-u_2B_3\beta_{hm}\right)\left(\frac{I_h}{N_h}S_m-\frac{I_h^*}{N_h}S_m-\frac{I_h}{N_h}S_m^*+\frac{I_h^*}{N_h}S_m^*\right)-\left(\mu_m+u_4\right)\left(I_m-I_m^*\right)\right\}$$

or

$$j_8=\left(\frac{I_m-I_m^*}{I_m}\right)\left\{\begin{array}{l}\left(\frac{I_h}{N_h}S_mB_3\beta_{hm}-\frac{I_h}{N_h}S_mu_2B_3\beta_{hm}-\frac{I_h^*}{N_h}S_mB_3\beta_{hm}+\frac{I_h^*}{N_h}S_mu_2B_3\beta_{hm}-\frac{I_h}{N_h}S_m^*B_3\beta_{hm}+\right.\\ \left.\frac{I_h}{N_h}S_m^*u_2B_3\beta_{hm}+\frac{I_h^*}{N_h}S_m^*B_3\beta_{hm}-\frac{I_h^*}{N_h}S_m^*u_2B_3\beta_{hm}\right)-\left(\mu_m+u_4\right)\left(I_m-I_m^*\right)\end{array}\right\}$$

or

$$j_8=\left\{\left(\frac{I_h}{N_h}S_mB_3\beta_{hm}\left(1-\frac{I_m^*}{I_m}\right)-\frac{I_h}{N_h}S_mu_2B_3\beta_{hm}\left(1-\frac{I_m^*}{I_m}\right)-\frac{I_h^*}{N_h}S_mB_3\beta_{hm}\left(1-\frac{I_m^*}{I_m}\right)+\frac{I_h^*}{N_h}S_mu_2B_3\beta_{hm}\left(1-\frac{I_m^*}{I_m}\right)\left(-\frac{I_h}{N_h}S_m^*B_3\beta_{hm}\times\right.\right.\right.$$

$$\left.\left.\left(1-\frac{I_m^*}{I_m}\right)+\frac{I_h}{N_h}S_m^*u_2B_3\beta_{hm}\left(1-\frac{I_m^*}{I_m}\right)\right)+\frac{I_h^*}{N_h}S_m^*B_3\beta_{hm}\left(1-\frac{I_m^*}{I_m}\right)-\frac{I_h^*}{N_h}S_m^*u_2B_3\beta_{hm}\left(1-\frac{I_m^*}{I_m}\right)\right)-(\mu_m+u_4)\frac{\left(I_m-I_m^*\right)^2}{I_m}\right\}$$

Consequently

$$j_8=\frac{I_h}{N_h}S_mB_3\beta_{hm}+\frac{I_h}{N_h}S_mu_2B_3\beta_{hm}\frac{I_m^*}{I_m}+\frac{I_h^*}{N_h}S_mB_3\beta_{hm}\frac{I_m^*}{I_m}+\frac{I_h^*}{N_h}S_mu_2B_3\beta_{hm}+\frac{I_h}{N_h}S_m^*B_3\beta_{hm}\frac{I_m^*}{I_m}+$$

$$\frac{I_h}{N_h}S_m^*u_2B_3\beta_{hm}+\frac{I_h^*}{N_h}S_m^*B_3\beta_{hm}+\frac{I_h^*}{N_h}S_m^*u_2B_3\beta_{hm}\frac{I_m^*}{I_m}-\frac{I_h}{N_h}S_mB_3\beta_{hm}\frac{I_m^*}{I_m}-\frac{I_h}{N_h}S_mu_2B_3\beta_{hm}$$

$$-\frac{I_h^*}{N_h}S_mB_3\beta_{hm}-\frac{I_h^*}{N_h}S_mu_2B_3\beta_{hm}\frac{I_m^*}{I_m}-\frac{I_h}{N_h}S_m^*B_3\beta_{hm}-\frac{I_h}{N_h}S_m^*u_2B_3\beta_{hm}\frac{I_m^*}{I_m}-\frac{I_h^*}{N_h}S_m^*B_3\beta_{hm}\frac{I_m^*}{I_m}$$

$$-\frac{I_h^*}{N_h}S_m^*u_2B_3\beta_{hm}-(\mu_m+u_4)\frac{\left(I_m-I_m^*\right)^2}{I_m}$$

Collecting positive and negative terms together in the system (4.40), we obtain

$$\frac{dV}{dt}=X-Y \tag{4.49}$$

where

$$X=\mu_hN_h+\pi\mu_hN_h\frac{S_{h_1}^*}{S_{h_1}}+\frac{B_1\beta_{mh}I_m^*}{N_h}\frac{\left(S_{h_1}-S_{h_1}^*\right)^2}{S_{h_1}}+\frac{u_2B_1\beta_{mh}I_m}{N_h}\frac{\left(S_{h_1}-S_{h_1}^*\right)^2}{S_{h_1}}+\theta_1R_h+\theta_1R_h^*\frac{S_{h_1}^*}{S_{h_1}}+u_1\theta_2S_{h_2}$$

$$+u_1\theta_2S_{h_2}^*\frac{S_{h_1}^*}{S_{h_1}}+\pi\mu_hN_h+u_2B_2\beta_{mh}\frac{I_m}{N_h}\frac{\left(S_{h_2}-S_{h_2}^*\right)^2}{S_{h_2}}+B_2\beta_{mh}\frac{I_m^*}{N_h}\frac{\left(S_{h_2}-S_{h_2}^*\right)^2}{S_{h_2}}B_1S_{h_1}\frac{\beta_{mh}I_m}{N_h}$$

$$+u_2B_1S_{h_1}\frac{\beta_{mh}I_m}{N_h}\frac{I_h^*}{I_h}+B_1S_{h_1}^*\frac{\beta_{mh}I_m}{N_h}\frac{I_h^*}{I_h}+u_2B_1S_{h_1}^*\frac{\beta_{mh}I_m}{N_h}+\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}u_2B_1S_{h_1}$$

$$+\frac{\beta_{mh}I_m^*}{N_h}B_1S_{h_1}^*+\frac{\beta_{mh}I_m^*}{N_h}u_2B_1S_{h_1}^*\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}+\frac{\beta_{mh}I_m}{N_h}u_2B_2S_{h_2}\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m}{N_h}B_2S_{h_2}^*\frac{I_h^*}{I_h}+$$

$$\frac{\beta_{mh}I_m}{N_h}u_2B_2S_{h_2}^*+\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}\frac{I_h^*}{I_h}+\frac{\beta_{mh}I_m^*}{N_h}u_2B_2S_{h_2}+\frac{\beta_{mh}I_m^*}{N_h}B_2S_{h_2}^*+\varphi I_m^*\frac{\left(A_m-A_m^*\right)^2}{kN_hA_m}+$$

$$u_5\eta_A \frac{\left(A_m - A_m^*\right)^2}{A_m} + \eta_A A_m + u_2 B_3 \beta_{hm} \frac{I_h}{N_h} \frac{\left(S_m - S_m^*\right)^2}{S_m} + B_3 \beta_{hm} \frac{I_h^*}{N_h} \frac{\left(S_m - S_m^*\right)^2}{S_m} + \eta_A u_5 A_m \frac{S_m^*}{S_m}$$

$$+ \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2}^* \frac{I_h^*}{I_h} + \eta_h I_h + \eta_h I_h^* \frac{T_h^*}{T_h} \delta_h T_h + \delta_h T_h^* \frac{R_h^*}{R_h} + \varphi S_m + \varphi S_m^* \frac{A_m^*}{A_m} + \varphi S_m^* \frac{\left(A_m - A_m^*\right)^2}{k N_h A_m}$$

$$+ \varphi I_m + \varphi I_m^* \frac{A_m^*}{A_m} + A_m^* \eta_A \frac{S_m^*}{S_m} + A_m^* \eta_A u_5 + \frac{I_h}{N_h} S_m B_3 \beta_{hm} + \frac{I_h}{N_h} S_m u_2 B_3 \beta_{hm} \frac{I_m^*}{I_m} + \frac{I_h^*}{N_h} S_m B_3 \beta_{hm} \frac{I_m^*}{I_m}$$

$$+ \frac{I_h^*}{N_h} S_m u_2 B_3 \beta_{hm} + \frac{I_h}{N_h} S_m^* B_3 \beta_{hm} \frac{I_m^*}{I_m} + \frac{I_h}{N_h} S_m^* u_2 B_3 \beta_{hm} + \frac{I_h^*}{N_h} S_m^* B_3 \beta_{hm} + \frac{I_h^*}{N_h} S_m^* u_2 B_3 \beta_{hm} \frac{I_m^*}{I_m}$$

$$Y = -\mu_h N_h \frac{S_{h_1}^*}{S_{h_1}} - \pi \mu_h N_h - \frac{B_1 \beta_{mh} I_m}{N_h} \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \frac{u_2 B_1 \beta_{mh} I_m^*}{N_h} \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \mu_h \frac{\left(S_{h_1} - S_{h_1}^*\right)^2}{S_{h_1}} - \theta_1 R_h^*$$

$$-\theta_1 R_h \frac{S_{h_1}^*}{S_{h_1}} - u_1 \theta_2 S_{h_2} \frac{S_{h_1}^*}{S_{h_1}} - u_1 \theta_2 S_{h_2}^* - \pi \mu_h N_h \frac{S_{h_2}^*}{S_{h_2}} - B_2 \beta_{mh} \frac{I_m}{N_h} \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - u_2 B_2 \beta_{mh}$$

$$\times \frac{I_m^* \left(S_{h_2} - S_{h_2}^*\right)^2}{N_h S_{h_2}} - \left(\mu_h + u_1 \theta_2\right) \frac{\left(S_{h_2} - S_{h_2}^*\right)^2}{S_{h_2}} - B_1 S_{h_1} \frac{\beta_{mh} I_m}{N_h} \frac{I_h^*}{I_h} - u_2 B_1 S_{h_1} \frac{\beta_{mh} I_m}{N_h} - B_1 S_{h_1}^* \frac{\beta_{mh} I_m}{N_h}$$

$$-u_2 B_1 S_{h_1}^* \frac{\beta_{mh} I_m}{N_h} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1} - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} B_1 S_{h_1}^* \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_1 S_{h_1}^*$$

$$-\frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2} - \frac{\beta_{mh} I_m}{N_h} B_2 S_{h_2}^* - \frac{\beta_{mh} I_m}{N_h} u_2 B_2 S_{h_2}^* \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}$$

$$-\frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2} \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} B_2 S_{h_2}^* \frac{I_h^*}{I_h} - \frac{\beta_{mh} I_m^*}{N_h} u_2 B_2 S_{h_2}^* - \left(\mu_h + \eta_h + a\right) \frac{\left(I_h - I_h^*\right)^2}{I_h} - \eta_h I_h \frac{T_h^*}{T_h} -$$

$$\eta_h I_h^* - \left(\mu_h + \delta_h\right) \frac{\left(T_h - T_h^*\right)^2}{T_h} - \delta_h T_h^* - \delta_h T_h \frac{R_h^*}{R_h} - \left(\mu_h + \theta_1\right) \frac{\left(R_h - R_h^*\right)^2}{R_h} - \varphi S_m \frac{A_m^*}{A_m} - \varphi S_m^* - \varphi S_m \frac{\left(A_m - A_m^*\right)^2}{k N_h A_m} -$$

$$\varphi I_m \frac{A_m^*}{A_m} - \varphi I_m^* - \varphi I_m \frac{\left(A_m - A_m^*\right)^2}{k N_h A_m} - \left(\mu_A + \eta_A + u_3\right) \frac{\left(A_m - A_m^*\right)^2}{A_m} - \eta_A A_m \frac{S_m^*}{S_m} - \eta_A u_5 A_m - A_m^* \eta_A -$$

$$A_m^* \eta_A u_5 \frac{S_m^*}{S_m} - B_3 \beta_{hm} \frac{I_h}{N_h} \frac{\left(S_m - S_m^*\right)^2}{S_m} - u_2 B_3 \beta_{hm} \frac{I_h^*}{N_h} \frac{\left(S_m - S_m^*\right)^2}{S_m} - \left(u_4 + \mu_m\right) \frac{\left(S_m - S_m^*\right)^2}{S_m} - \frac{I_h}{N_h} S_m B_3 \beta_{hm} \frac{I_m^*}{I_m}$$

$$-\frac{I_h}{N_h} S_m u_2 B_3 \beta_{hm} - \frac{I_h^*}{N_h} S_m B_3 \beta_{hm} - \frac{I_h^*}{N_h} S_m u_2 B_3 \beta_{hm} \frac{I_m^*}{I_m} - \frac{I_h}{N_h} S_m^* B_3 \beta_{hm} - \frac{I_h}{N_h} S_m^* u_2 B_3 \beta_{hm} \frac{I_m^*}{I_m} -$$

$$\frac{I_h^*}{N_h}S_m^*B_3\beta_{hm}\frac{I_m^*}{I_m}-\frac{I_h^*}{N_h}S_m^*u_2B_3\beta_{hm}-\left(\mu_m+u_4\right)\frac{\left(I_m-I_m^*\right)^2}{I_m}$$

Thus from equation (4.49), if $\mathrm{X}<\mathrm{Y}$ then $\frac{dV}{dt}$ will be negative definite, meaning that $\frac{dV}{dt}<0$. It follows that $\frac{dV}{dt}=0$ if and only if $S_{h_1}=S_{h_1}^*$, $S_{h_2}=S_{h_2}^*$, $I_h=I_h^*$, $T_h=T_h^*$, $R_h=R_h^*$, $A_m=A_m^*$, $S_m=S_m^*$ and $I_m=I_m^*$. Therefore the largest compact invariant set in $\left\{S_{h_1}^*,S_{h_2}^*,I_h^*,T_h^*,R_h^*,A_m^*,S_m^*,I_m^*\in\Omega:\ \frac{dV}{dt}=0\right\}$ is the singleton $\left\{E^*\right\}$ where E^* is the endemic equilibrium of the model system (4.1). By LaSalle's invariant principle, then it is implies that E^* is globally asymptotically stable in Ω if $\mathrm{X}<\mathrm{Y}$.This completes the proof.

4.9 Numerical Simulations

4.9.1 Numerical Simulations and Discussion of Results.

In this chapter, we illustrate the analytical results of the study by carrying out numerical simulations of the model system (4.1) using the set of estimated parameter values given in Table 3.4.

Figures 4.2 show the distribution of population with time in all classes of human and mosquito when no control is applied.

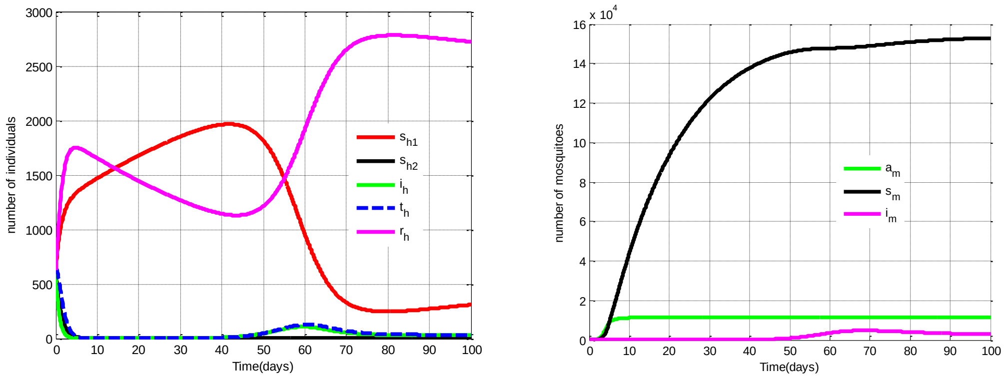


Figure 4. 2: Distribution of population with time in all classes of human and mosquito when no control is applied that is $u_1 = 1$**,** $u_2 = 0$**,** $u_3 = 0$**,** $u_4 = 0$**,** $u_5 = 0$**,** $\delta = 1$

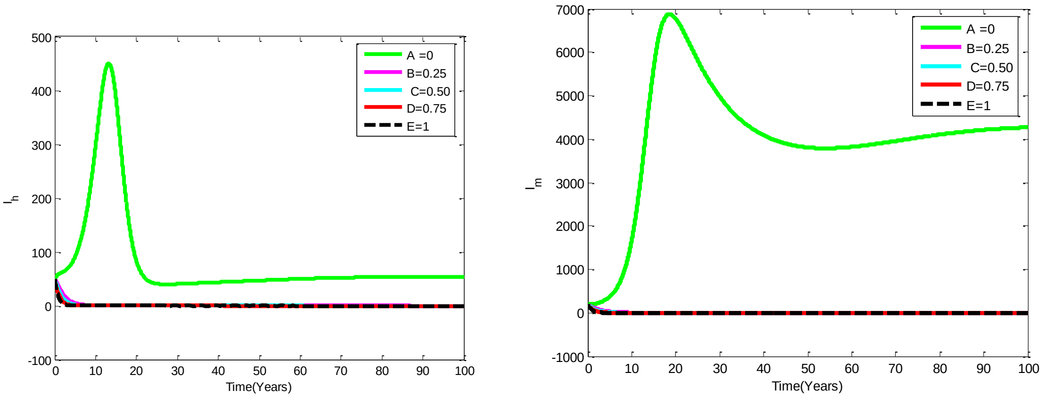
From Figure 4. 2, it is observed that the human infection reaches a peak between the 45th and the 75th day .The infection of the mosquitoes delayed. The total number of infected humans obtained from System (4.1) is lower than observations in Tanzania. The difference is due to the absence of the data in the whole country of Tanzania (Rodrigues *et al.*, 2012).

Figures 4.3 (i)-(ii) shows the variation of infected human and mosquito populations with combine use of all five controls:

$A = u_1 = u_2 = u_3 = u_4 = u_5 = 0$, $B = u_1 = u_2 = u_3 = u_4 = u_5 = 0.25$,

$C = u_1 = u_2 = u_3 = u_4 = u_5 = 0.50$, $D = u_1 = u_2 = u_3 = u_4 = u_5 = 0.75$,

$E = u_1 = u_2 = u_3 = u_4 = u_5 = 1$

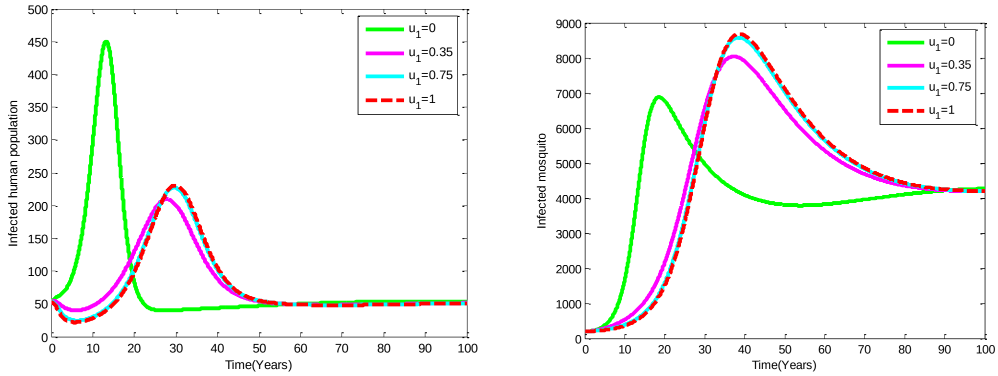


(i) (ii)

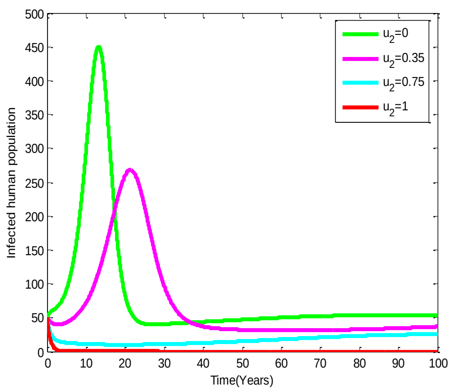
Figure 4.3: (i)-(ii): Variation of infected human and mosquito populations with combine use of all five controls.

From figure 4.3 (i)-(ii), it is observed that when all the controls are used, the disease is eradicated.

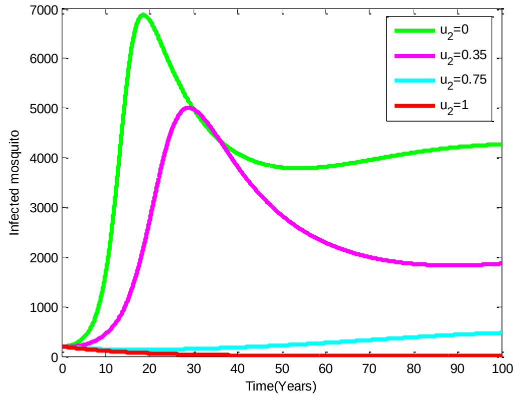
Figures 4.4(i)-(x) show the variation of infected human and infected mosquito populations with different levels of campaign of educating Careless human susceptible u_1, vector human contact u_2, reducing vector breeding areas u_3, insecticide application u_4 and maturation rate from larvae to adult u_5.



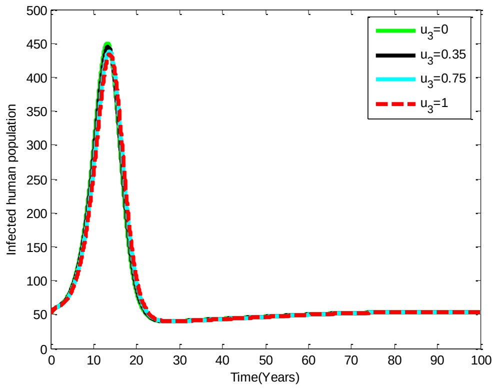
(i) (ii)



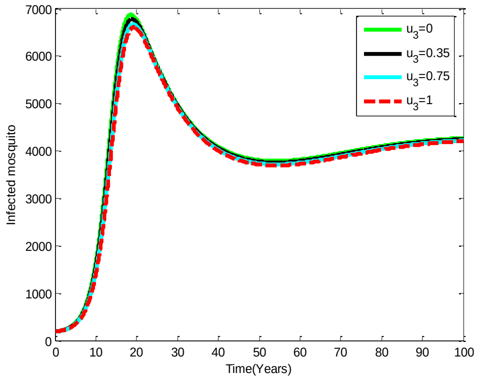
(iii)



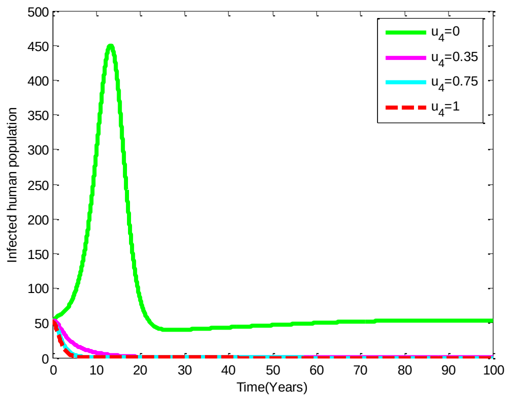
(iv)



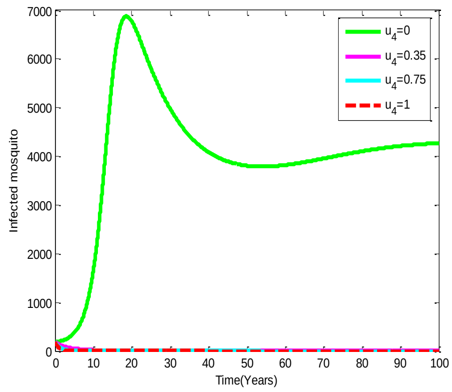
(v)



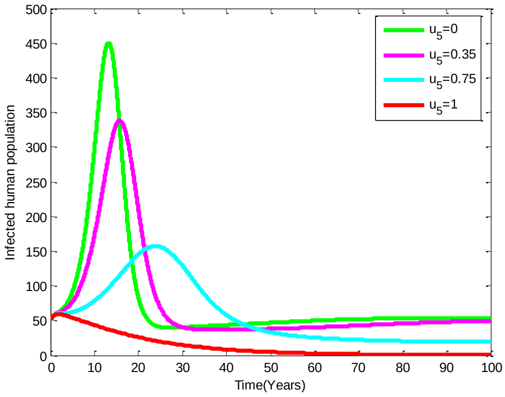
(vi)



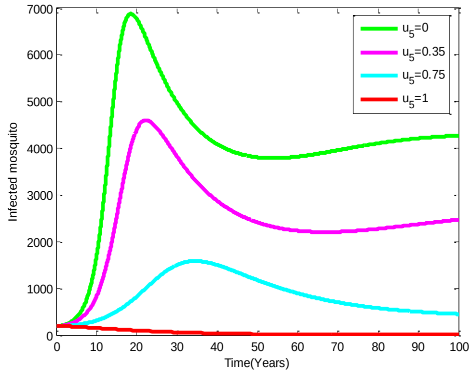
(vii)



(viii)



(ix)



(x)

Figure 4.4: (i)-(x):Variation of infected human and infected mosquito populations with different levels of campaign of educating Careless human susceptible , vector human contact , reducing vector breeding areas , insecticide application and maturation rate from larvae to adult.

From Figure 4.4 (i)-(x) shows that, the increase of control reduce the disease.

4.10 Optimal Control Analysis

In this section, we adopt approach similar to that of Seidu and Makinde, (2014). The aim is to seek optimal levels of the control strategies that is u_1, u_2, u_3, u_4 and u_5 needed to minimize the number of infected human and the cost of implementing the control strategies. A functional J given by

$$J = \min_{u_i, i\in[1,5]} \int_0^T \left(A_1 S_{h_1}(t) + A_2 S_{h_2}(t) + A_3 I_h(t) + A_4 I_m(t) + \frac{1}{2}D_1u_1^2 + \frac{1}{2}D_2u_2^2 + \frac{1}{2}D_3u_3^2 + \frac{1}{2}D_4u_4^2 + \frac{1}{2}D_5u_5^2 \right) dt \quad (4.50)$$

subject to the differential equations (4.1) and initial condition chosen as $S_{h_1}(0)\geq 0$, $S_{h_2}(0)\geq 0$, $I_h(0)\geq 0$, $T_h(0)\geq 0$, $R_h(0)\geq 0$, $A_m(0)\geq 0$, $S_m(0)\geq 0$ and $I_m(0)\geq 0$

where the $D_i s$ are positive weights which measure relative costs of implementing the respective control strategies over the period [0,T], while the terms $\frac{D_i u_i^2}{2}$ measure the cost of the control strategies, A_i is the positive weights, $S_{h_1}(t)$ is careful human Susceptibles, $S_{h_2}(t)$ is careless human Susceptibles, I_h is the infected human, I_m is the infected mosquito and T is the final time.

Quadratic cost on the controls is preferred and this is similar with what is in other literature on epidemic controls (Makinde and Okosun, 2011). The goal is to minimize infection, while minimizing the cost of control.

Thus, we seek an optimal control u^* such that

$$J(u^*) = \min\left\{J(u) \mid u \in U\right\} \tag{4.51}$$

where U is the control set defined by $U = \{(u_1, \ldots\ldots\ldots, u_5)$ such that u_i are measurable with $0 \le u_i(t) \le 1;\ \forall t \in [0,T]\}$ is the set of admissible controls. The necessary conditions that an optimal must satisfy come from Pontryagin's Maximum (Pontryagin *et al.*, 1962). This principle converts a dynamic system (differential equation of the i^{th} state variable) and (4.50) above into a problem of minimizing pointwise a Hamiltonian H, with respect to u where;

$$H = A_1 S_{h_1} + A_2 S_{h_2} + A_3 I_h + A_4 I_m + \frac{1}{2}D_1 u_1^2 + \frac{1}{2}D_2 u_2^2 + \frac{1}{2}D_3 u_3^2 + \frac{1}{2}D_4 u_4^2 + \frac{1}{2}D_5 u_5^2$$

$$+\lambda_1\left[(1-\pi)\mu_h N_h - (1-u_2)B_1\beta_{mh}\frac{I_m}{N_h}S_{h_1} - \mu_h S_{h_1} + \theta_1 R_h + u_1\theta_2 S_{h_2}\right]$$

$$+\lambda_2\left[\pi\mu_h N_h - (1-u_2)B_2\beta_{mh}\frac{I_m}{N_h}S_{h_2} - \mu_h S_{h_2} - u_1\theta_2 S_{h_2}\right]$$

$$
\begin{aligned}
&+\lambda_3\left[\left((1-u_2)B_1S_{h_1}+(1-u_2)B_2S_{h_2}\right)\beta_{mh}\frac{I_m}{N_h}-(\mu_h+\eta_h+a)I_h\right] \\
&+\lambda_4\left[\eta_h I_h-(\mu_h+\delta_h)T_h\right]+\lambda_5\left[\delta_h T_h-(\mu_h+\theta_1)R_h\right] \\
&+\lambda_6\left[\varphi\left(1-\frac{A_m}{kN_h}\right)(S_m+I_m)-\mu_A A_m-u_3A_m-(1-u_5)\eta_A A_m\right] \\
&+\lambda_7\left[(1-u_5)\eta_A A_m-\left((1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}+\mu_m\right)S_m-u_4S_m\right] \\
&+\lambda_8\left[(1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}S_m-(\mu_m+u_4)I_m\right]
\end{aligned}
\qquad (4.52)
$$

where $\lambda_i\,(i=1,2,...,8)$ are the adjoint variables or co-state variables which determine the adjoint system, together with the state system (4.1) describes the optimality system. Pontryagin's Maximum principle (Pontryagin *et al*., 1962) and the existence result for optimal control from Fleming and Rishel, (1975) can be used to obtain the following proposition:

Proposition 1. The optimal control 5-tuple (u_1,u_2,u_3,u_4,u_5) minimizes the functional J if there exist adjoint variables $\lambda_i, i=1,2,.....,8$ that satisfy the adjoint system given by

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial S_{h_1}}.$$

From (4.52),

$$\frac{\partial H}{\partial S_{h_1}}=A_1+\lambda_1\left[-(1-u_2)B_1\beta_{mh}\frac{I_m}{N_h}-\mu_h\right]+\lambda_3\left[(1-u_2)B_1\right]\beta_{mh}\frac{I_m}{N_h}.$$

Hence

$$\frac{d\lambda_1}{dt} = -A_1 + \lambda_1\left[(1-u_2)B_1\beta_{mh}\frac{I_m}{N_h} + \mu_h\right] - \lambda_3\left[(1-u_2)B_1\right]\beta_{mh}\frac{I_m}{N_h}$$

Others will be obtained using the same method.

Therefore

$$\frac{d\lambda_2}{dt} = -A_2 - \lambda_1[u_1\theta_2] + \lambda_2\left[(1-u_2)B_2\beta_{mh}\frac{I_m}{N_h} + \mu_h + u_1\theta_2\right] - \lambda_3\left[(1-u_2)B_2\beta_{mh}\frac{I_m}{N_h}\right]$$

$$\frac{d\lambda_3}{dt} = -A_3 + \lambda_3[\mu_h + \eta_h + a] - \lambda_4\eta_h + \lambda_7\left[(1-u_2)B_3\beta_{hm}\frac{S_m}{N_h}\right] - \lambda_8\left[(1-u_2)B_3\beta_{hm}\frac{S_m}{N_h}\right]$$

$$\frac{d\lambda_4}{dt} = \lambda_4(\mu_h + \delta_h) - \lambda_5\delta_h$$

$$\frac{d\lambda_5}{dt} = -\lambda_1\theta_1 + \lambda_5(\mu_h + \theta_1)$$

$$\frac{d\lambda_6}{dt} = \lambda_6\left[\frac{\varphi}{kN_h}(S_m + I_m) + \left(\mu_A + u_3 + (1-u_5)\eta_A\right)\right] - \lambda_7(1-u_5)\eta_A \tag{4.53}$$

$$\frac{d\lambda_7}{dt} = -\lambda_6\left[\varphi\left(1 - \frac{A_m}{kN_h}\right)\right] + \lambda_7\left[\left((1-u_2)B_3\beta_{hm}\frac{I_h}{N_h} + \mu_m\right) + u_4\right] - \lambda_8\left[(1-u_2)B_3\beta_{hm}\frac{I_h}{N_h}\right]$$

$$\frac{d\lambda_8}{dt} = -A_4 + \lambda_1\left[(1-u_2)B_1\beta_{mh}\frac{S_{h_1}}{N_h}\right] + \lambda_2\left[(1-u_2)B_2\beta_{mh}\frac{S_{h_2}}{N_h}\right]$$

$$-\lambda_3\left[\left((1-u_2)B_1S_{h_1} + (1-u_2)B_2S_{h_2}\right)\frac{\beta_{mh}}{N_h}\right] - \lambda_6\left[\varphi\left(1 - \frac{A_m}{kN_h}\right)\right] + \lambda_8(\mu_m + u_4)$$

With transversality conditions

$$\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = \lambda_4(t) = \lambda_5(t) = \lambda_6(t) = \lambda_7(t) = \lambda_8(t) = 0$$

To get the characterization of the optimal control we solve the equations

$$\frac{\partial H}{\partial u_i} = 0 \text{ at } u_i = u_i^*$$

where $i = 1, 2, ...n$ and n is number of controls.

The first control is obtained as

$$\frac{\partial H}{\partial u_1} = D_1 u_1 + \lambda_1 \left[\theta_2 S_{h_2} \right] - \lambda_2 \left[\theta_2 S_{h_2} \right] \text{ from } \quad (4.52)$$

Then we set $\frac{\partial H}{\partial u_1} = 0$ to get

$$D_1 u_1 + \lambda_1 \left[\theta_2 S_{h_2} \right] - \lambda_2 \left[\theta_2 S_{h_2} \right] = 0$$

or

$$D_1 u_1 = -\theta_2 S_{h_2} \lambda_1 + \theta_2 S_{h_2} \lambda_2 .$$

Consequently

$$u_1 = \frac{-\theta_2 S_{h_2} \lambda_1 + \theta_2 S_{h_2} \lambda_2}{D_1} .$$

Other controls for u_2, u_3, u_4 and u_5 are obtained similarly. Thus

$$u_2 = \left\{ \begin{array}{l} \lambda_3 \left[\left(B_1 S_{h_1} + B_2 S_{h_2} \right) \beta_{mh} \frac{I_m}{N_h} \right] + \lambda_8 \left[B_3 \beta_{hm} \frac{I_h}{N_h} S_m \right] - \\ \lambda_1 \left[B_1 \beta_{mh} \frac{I_m}{N_h} S_{h_1} \right] - \lambda_2 \left[B_2 \beta_{mh} \frac{I_m}{N_h} S_{h_2} \right] - \lambda_7 \left[B_3 \beta_{hm} \frac{I_h}{N_h} S_m \right] \end{array} \right\} \Bigg/ D_2 , \qquad u_3 = \frac{\lambda_6 A_m}{D_3} ,$$

$$u_4 = \frac{\lambda_7 S_m + \lambda_8 I_m}{D_4} \quad \text{and} \quad u_5 = \frac{\eta_A A_m \left(-\lambda_6 + \lambda_7 \right)}{D_5}$$

In order to satisfy the given bounds for the control functions, that is $0 \le u(t) \le 1$ and $t \in \left[0, T \right]$ the optimal control is restricted to $u_i^* = \min \left\{ 1, \max \left(0, u_i \right) \right\}$. Therefore

$$u_1^* = \min \left\{ 1, \max \left(0, \frac{-\theta_2 S_{h_2} \lambda_1 + \theta_2 S_{h_2} \lambda_2}{D_1} \right) \right\}$$

$$u_2^* = \min \left\{ 1, \max \left(0, \left\{ \begin{array}{l} \lambda_3 \left[\left(B_1 S_{h_1} + B_2 S_{h_2} \right) \beta_{mh} \frac{I_m}{N_h} \right] + \lambda_8 \left[B_3 \beta_{hm} \frac{I_h}{N_h} S_m \right] - \\ \lambda_1 \left[B_1 \beta_{mh} \frac{I_m}{N_h} S_{h_1} \right] - \lambda_2 \left[B_2 \beta_{mh} \frac{I_m}{N_h} S_{h_2} \right] - \lambda_7 \left[B_3 \beta_{hm} \frac{I_h}{N_h} S_m \right] \end{array} \right\} \Bigg/ D_2 \right) \right\}$$

$$u_3^* = \min\left\{1, \max\left(0, \frac{\lambda_6 A_m}{D_3}\right)\right\},$$ (4.54)

$$u_4^* = \min\left\{1, \max\left(0, \frac{\lambda_7 S_m + \lambda_8 I_m}{D_4}\right)\right\}$$

$$u_5^* = \min\left\{1, \max\left(0, \frac{\eta_A A_m\left(-\lambda_6 + \lambda_7\right)}{D_5}\right)\right\}.$$

4.10.1 Existence of the Optimal Controls

Proof:

The existence of the optimal controls is obtained from Fleming and Rishel, (1975) due to the convexity of the integrand of the functional J with respect to the 5-tuple $\left(u_1, u_2, u_3, u_4, u_5\right)$ minimizes the functional J a prior boundedness of the state solutions, and the Lipschitz property of the state system (4.1) with respect to the state variables $\left(S_{h_1}, S_{h_2}, I_h, T_h, R_h, A_m, S_m, I_m\right)$. Using Pontryagin's Maximum Principle, the costate or adjoint equations (4.53) are obtained by differentiating the Hamiltonian Partially with respect to the state variables. Hence, we have $\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S_{h_1}}$,

$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial S_{h_2}}$, $\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I_h}$, $\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial T_h}$, $\frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial R_h}$, $\frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial A_m}$,

$\frac{d\lambda_7}{dt} = -\frac{\partial H}{\partial S_m}$ and $\frac{d\lambda_7}{dt} = -\frac{\partial H}{\partial I_m}$.

With transversality conditions

$$\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = \lambda_4(t) = \lambda_5(t) = \lambda_6(t) = \lambda_7(t) = \lambda_8(t) = 0$$

Since the Hamiltonian is minimized at the optimal controls, the optimality conditions $\frac{\partial H}{\partial u_i} = 0$ at $u_i = u_i^*$ are met. These optimality conditions can be used to obtain expressions for u_i^* by standard control arguments involving the bounds on the controls, (4.54) is obtained. This completes the proof.

4.10.2 Effects of Optimal Control Strategies

In this section, we illustrate the analytical results of the study by carrying out numerical simulations of the model system (4.1) and study the effects of campaign to educate the careless human susceptible (u_1), control vector-human contact (u_2), removing vector breeding areas (u_3), insecticides application (u_4) and control maturation rate from larvae to adult (u_5). We investigate and compare numerical results in the following strategies:

(i) when all controls are set to zero, (ii) when only control vector-human contact (u_2) is used to optimize the objective function (J) while other controls are set to zero, (iii) when insecticides application (u_4) is used to optimize the objective function (J) while other controls are set to zero, (iv) when campaign to educate the careless human susceptible (u_1) and control vector-human contact (u_2) are used to optimize the objective function (J) while other controls are set to zero, (v)When removing vector breeding areas (u_3), insecticides application (u_4) and control maturation rate from larvae to adult (u_5) are used to optimize the objective function

(J) while other controls are set to zero, (vi) when removing vector breeding areas (u_3) and control maturation rate from larvae to adult (u_5) are used to optimize the objective function (J) while other controls are set to zero, (vii) when control vector-human contact (u_2) and insecticides application (u_4) are used to optimize the objective function (J) while other controls are set to zero and (viii) when all controls are used to optimize the objective function (J). Parameter values are obtained from the model fitting in table 3.4.

Figures 4.5 (i)-(viii) show the effects of optimal control strategies on the spread of dengue fever disease in the population.

The figures show the effects of optimal control of the model system (4.1) using the parameter values in Table 3.4 for different strategies as shown below.

1. $u_1 = u_2 = u_3 = u_4 = u_5 = 0$

2. $u_2 \neq 0, u_1 = u_3 = u_4 = u_5 = 0$

3. $u_4 \neq 0, u_1 = u_2 = u_3 = u_5 = 0$

4. $u_1 \neq 0, u_2 \neq 0, u_3 = u_4 = u_5 = 0$

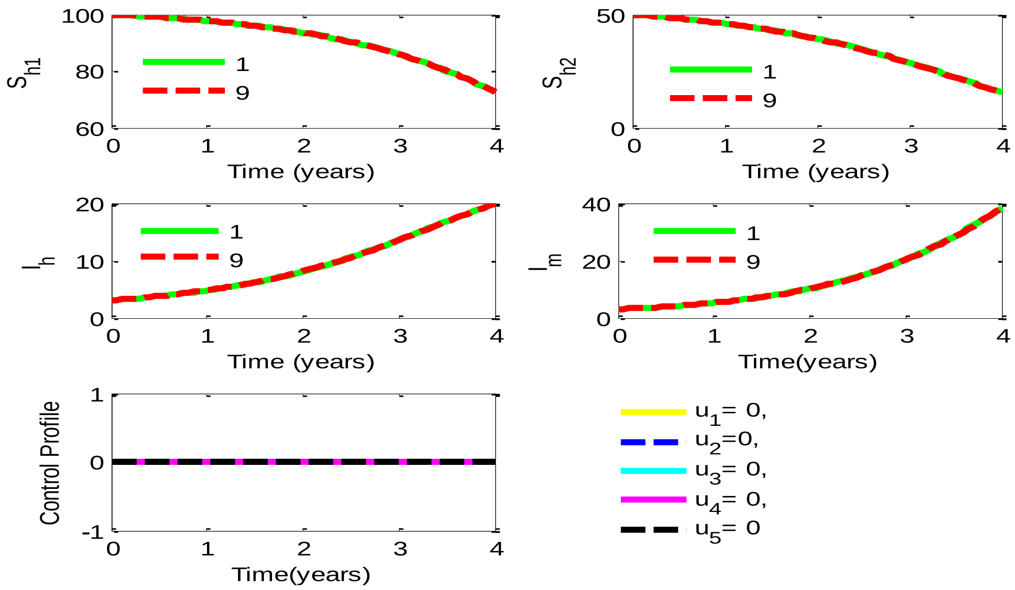
5. $u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 \neq 0, u_5 \neq 0$

6. $u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 = 0, u_5 \neq 0$

7. $u_2 \neq 0, u_4 \neq 0, u_1 = u_3 = u_5 = 0$

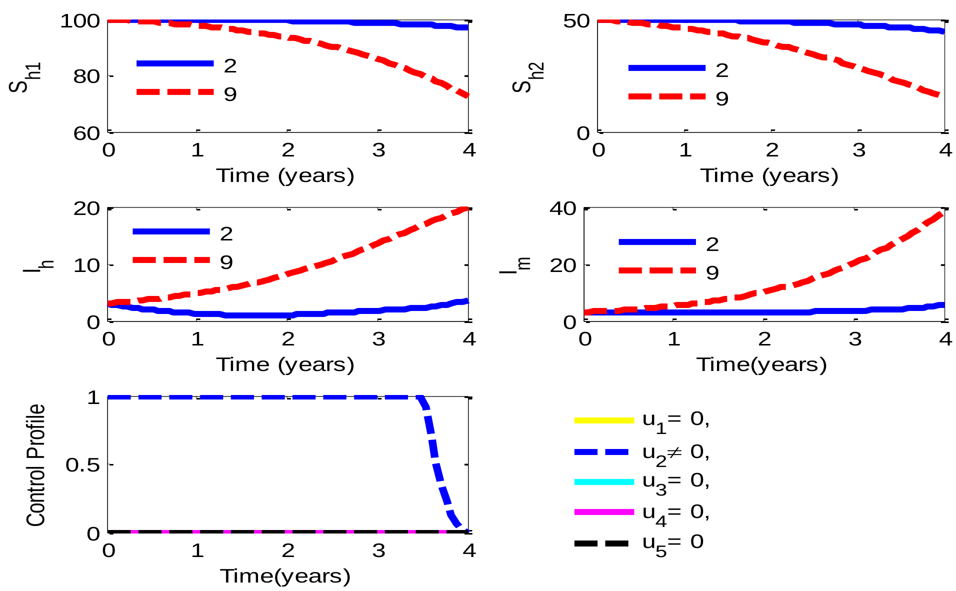
8. $u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0, u_5 \neq 0$

9. $u_1 = u_2 = u_3 = u_4 = u_5 = 0$



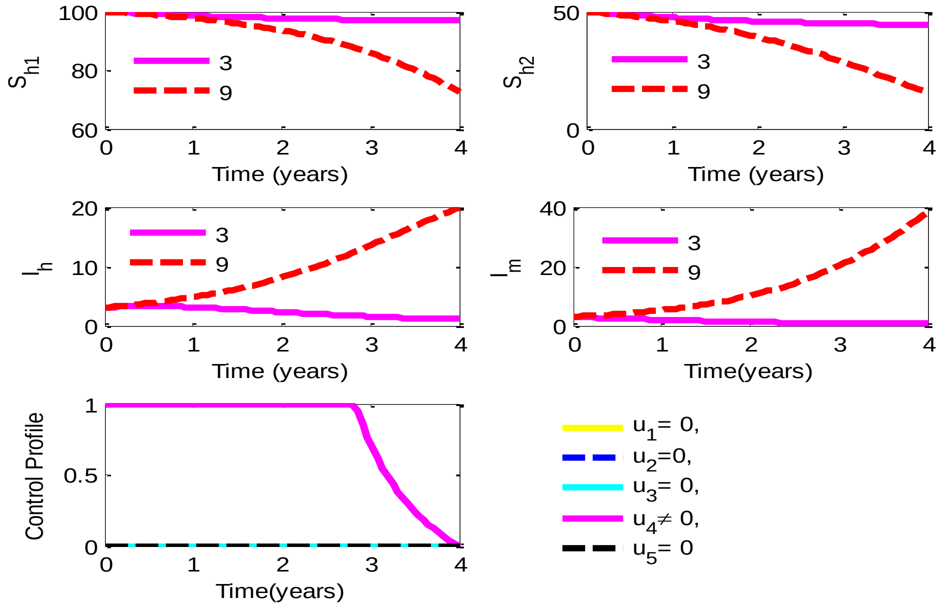
S_h1
S_h2
I_h
I_m
Time (years)
Time(years)
Control Profile
1
9
u_1= 0,
u_2=0,
u_3= 0,
u_4= 0,
u_5= 0

(i)



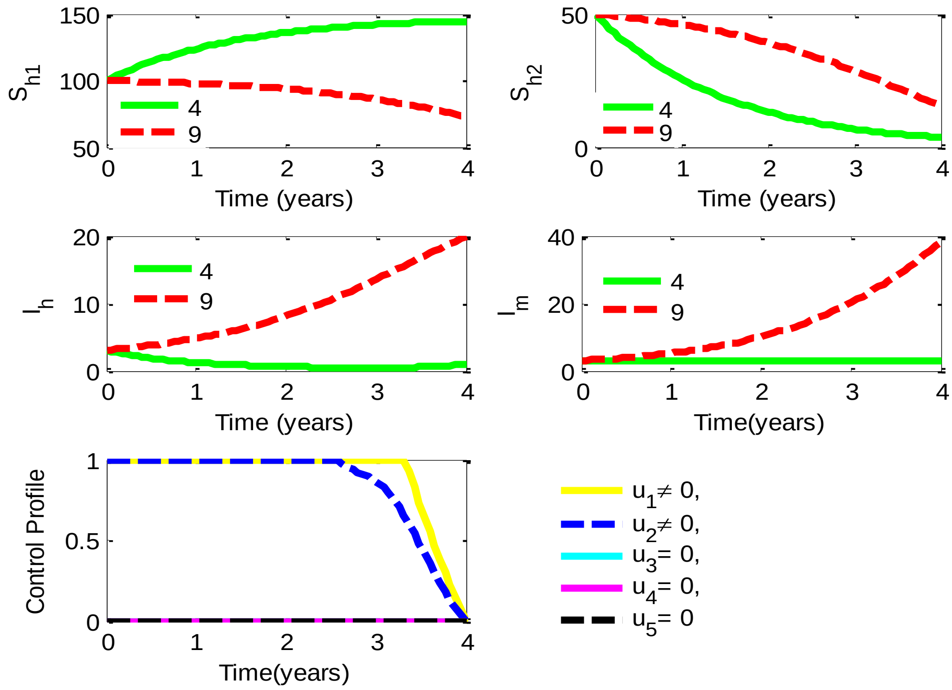
S_h1
S_h2
I_h
I_m
Time (years)
Time(years)
Control Profile
2
9
u_1= 0,
u_2≠ 0,
u_3= 0,
u_4= 0,
u_5= 0

(ii)



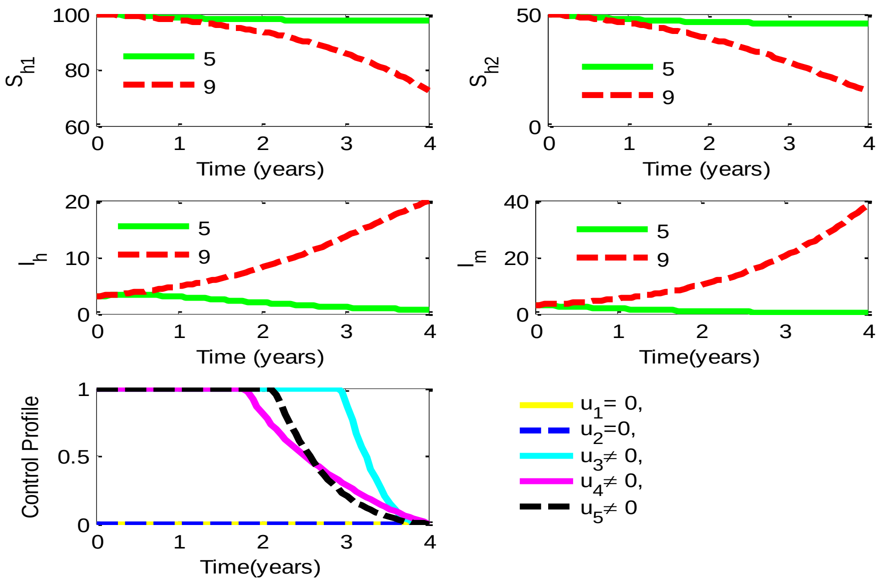
S_h1
Time (years)
S_h2
Time (years)
I_h
Time (years)
I_m
Time(years)
Control Profile
Time(years)
3
9
u_1= 0,
u_2=0,
u_3= 0,
u_4≠ 0,
u_5= 0

(iii)



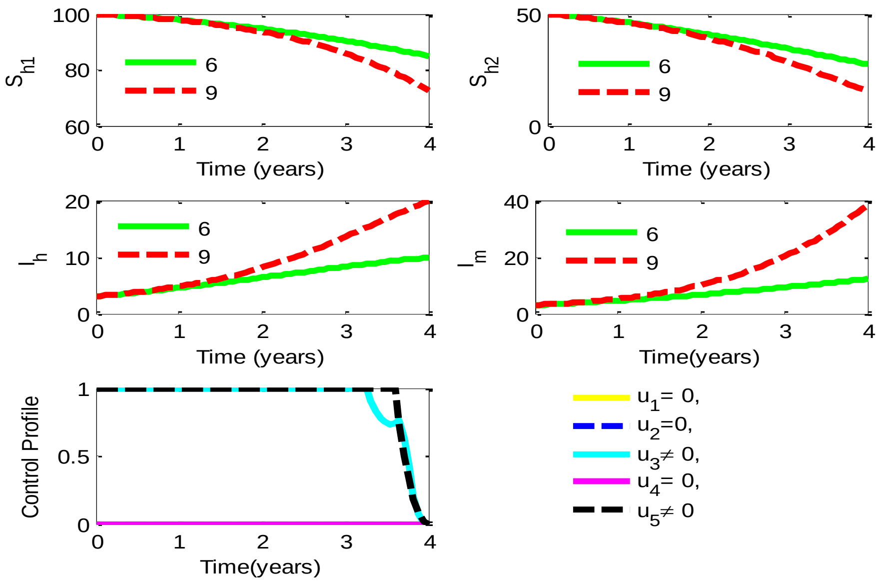
S_h1
Time (years)
S_h2
Time (years)
I_h
Time (years)
I_m
Time(years)
Control Profile
Time(years)
4
9
u_1≠ 0,
u_2≠ 0,
u_3= 0,
u_4= 0,
u_5= 0

(iv)



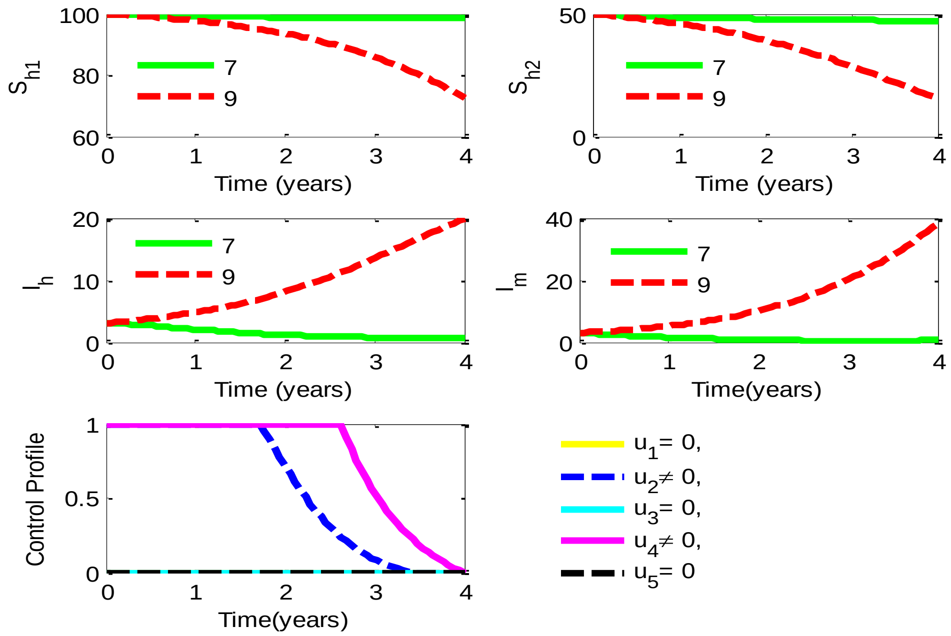
S_h1
S_h2
Time (years)
I_h
I_m
Time(years)
Control Profile
5
9
u_1= 0,
u_2=0,
u_3≠ 0,
u_4≠ 0,
u_5≠ 0

(v)

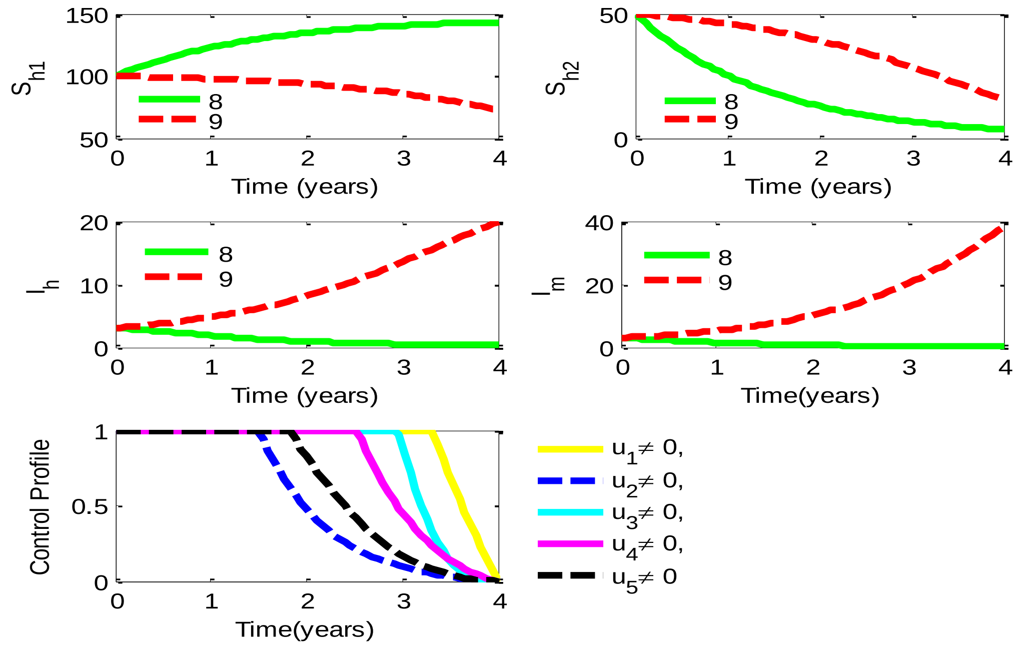


S_h1
S_h2
Time (years)
I_h
I_m
Time(years)
Control Profile
6
9
u_1= 0,
u_2=0,
u_3≠ 0,
u_4= 0,
u_5≠ 0

(vi)



(vii)



(viii)

Figures 4.5: (i)-(viii): Simulations of model system (4.1) showing the effects of optimal control strategies on the spread of dengue fever disease in the population.

From figure 4.5(i), it is observed that when no control is applied careful and careless human susceptible population decrease and infected human and mosquito populations increase. The control profile shows that all controls are in a lower bound. From Figures 4.5(ii), (iii), (v), (vi) and (vii), it is observed that careful and careless human susceptible population increased while infected human and mosquito population decreased.

From Figures 4.5 (iv) and (viii) it is seen that careful susceptible increase, careless susceptible decrease due to individual getting education and move to careful susceptible while infected human and mosquito population decreases.

Control profile for Figure4.5 (ii) shows that control vector-human contact (u_2) is at upper bound for 3.486 years before dropping down to the lower bound while u_1, u_3, u_4 and u_5 are at lower bound till the final time.

The control profile for Figure 4.5 (iii), insecticides application (u_4) is at upper bound for 2.8 years before dropping down to the lower bound while u_1, u_2, u_3 and u_5 are maintained at the lower bound till the final time.

Control profile for Figure 4.5 (iv) shows that Campaign for educating careless human susceptible (u_1) is at upper bound for 3.314 years and control vector-human contact (u_2) is at 2.514 years before dropping down to the lower bound while u_3, u_4 and u_5 are at lower bound till the final time.

Control profile for Figure 4.5 (v) shows that removing vector breeding areas (u_3) is at upper bound for 2.914 years, insecticides application (u_4) is at upper bound for 1.771years and control maturation rate from larvae to adult (u_5) is at upper bound for 2.114 years before dropping down to the lower bound, while u_1 and u_2 remains at the lower bound till the final time.

Control profile for Figure 4.5(vi),shows that control maturation rate from larvae to adult (u_5) are at upper bound for 3.6 years and removing vector breeding areas (u_3) is at upper bound for 3.257 years before dropping down to the lower bound while u_1 , u_2 and u_3 are maintained at the lower bound till the final time.

Control Profile for Figure 4.5 (vii), shows that control vector-human contact (u_2) is at upper bound for 1.717 years and insecticides application u_4 is at upper bound for 2.571 years before dropping down to the lower bound while u_1, u_3 and u_5 are maintained at the lower bound till the final time.

Control profile for Figure 4.5 (viii) shows that Campaign for educating careless human susceptible (u_1) is at upper bound for 3.257 years, control vector-human contact (u_2) is at upper bound for 1.486 years, removing vector breeding areas (u_3) is at upper bound for 2.914 years, insecticides application (u_4) is at upper bound for

2.514 years and control maturation rate from larvae to adult (u_5) is at upper bound for 1.771 years before dropping down to the lower bound.

4.11 Summary

In this subsection, optimal control analysis for dengue fever disease model was performed using Pontryagin's maximum principle. Conditions for optimal control of the disease were derived and analysed with an effective use of campaign to educate the careless human susceptible (u_1), control vector-human contact (u_2), removing vector breeding areas (u_3), insecticides application (u_4) and control maturation rate from larvae to adult (u_5). It is observed that from Figures 4.5 (ii)-(viii) infected human and mosquito populations are decreased, indicating that on application of any control or a combination of controls decrease the growth rate of dengue fever disease in Tanzania. However there is cost associated with this control, implying the need of minimizing the costs as financial resources are always scarce. In doing this, Cost effectiveness analysis is preferred in order to find the best strategy to fight the disease and minimize the cost.

CHAPTER FIVE

COST EFFECTIVENESS ANALYSIS

Cost effectiveness analysis is used to determine the most cost effective strategy to use to control the disease. To achieve this purpose we need to compare the differences between the costs and health outcomes of these interventions. This is done by calculating the incremental cost-effectiveness ratio (ICER) which is generally described as the additional cost per additional health outcome. When comparing two or more competing intervention strategies incrementally, one intervention should be compared with the next-less-effective alternative (Okosun *et al.*, 2013).

The ICER numerator includes the differences in intervention costs, averted disease costs, costs of prevented cases and averted productivity losses if applicable. The ICER denominator is the differences in health outcomes (Okosun *et al.*, 2013).

We rank the strategies in increasing order of effectiveness, namely (1) when all controls are set to zero, (2) When removing vector breeding areas (u_3) and control maturation rate from larvae to adult $(u_5)$is used to optimize the objective function (J) while other controls are set to zero, (3)When insecticides application (u_4) is used to optimize the objective function (J)while other controls are set to zero, (4) When removing vector breeding areas(u_3), insecticides application (u_4) and control maturation rate from larvae to adult (u_5) are used to optimize the objective function

(J) while campaign for educating the careless human susceptible (u_1) and control vector human contact $(u_2)$are set to zero, (5) When control vector-human contact (u_2) and insecticides application (u_4) are used to optimize the objective function (J) while other controls are set to zero,(6) When all controls are used to optimize the objective function (J) , (7) When only control vector human contact (u_2) is used to optimize the objective function (J) while other controls are set to zero and (8)When campaign to educate the careless human susceptible(u_1) and control vector-human contact(u_2) are used to optimize the objective function (J)while other controls are set to zero.

The difference between the total infectious individuals without control and the total infectious individuals with control is used to determine the ‘‘total number of infections averted’’ as shown in table 5.1 of cost-effectiveness analysis (Okosun *et al.*, 2013)

Table 5. 1: Ranking of Control strategies in order of increasing total infection averted

Strategy	Control	Total infection averted	Total cost ($)	J
Strategy 1	$u_1 = u_2 = u_3 = u_4 = u_5 = 0$	0.0000	0	4280.5
Strategy 2	$u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 = 0, u_5 \neq 0$	0.0583	0.3716	3298.2
Strategy 3	$u_4 \neq 0, u_1 = u_2 = u_3 = u_5 = 0$	0.1401	2.9681	3712.9
Strategy 4	$u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 \neq 0, u_5 \neq 0$	0.1457	2.5303	3756.3
Strategy 5	$u_2 \neq 0, u_4 \neq 0, u_1 = u_3 = u_5 = 0$	0.1655	3.7126	3927.5
Strategy 6	$u_1 \neq u_2 \neq u_3 \neq u_4 \neq u_5 \neq 0$	0.1697	6.7670	1248.9
Strategy 7	$u_2 \neq 0, u_1 = u_3 = u_4 = u_5 = 0$	0.1709	2.5798	4086.7
Strategy 8	$u_1 \neq 0, u_2 \neq 0, u_3 = u_4 = u_5 = 0$	0.1813	6.0612	1417.0

From Table 5.1 shows that strategy 8 is the best, as it avert majority, but in this study it intend to find strategy which minimize disease and minimize the cost. This is done by comparing strategy 2, 3, 4, 5, 6, 7 and 8 as shown below:

Table 5. 2: (i): we eliminate strategy 1 and compare strategy 2 and 3

Strategy	Control	Total infection averted	Total cost ($)	J
Strategy 2	$u_1=0, u_2=0, u_3\neq 0, u_4=0, u_5\neq 0$	0.0583	0.3716	3298.2
Strategy 3	$u_4\neq 0, u_1=u_2=u_3=u_5=0$	0.1401	2.9681	3712.9

Then we calculate ICER as follows

$$\text{ICER } 2=\frac{0.3716}{0.0583}=6.373927959 \quad \text{and} \quad \text{ICER } 3=\frac{2.9681-0.3716}{0.1401-0.0583}=31.74205379$$

The comparison between ICER (2) and ICER (3) shows a cost saving of $\$6.373927959$ for strategy 2 over strategy 3. The ICER for strategy 2 indicates that the strategy 3 is expensive and less effective. Therefore we exclude strategy 3 so that it does not consume limited resources.

Table 5.2 (ii): We compare strategies 2 and 4.

Strategy	Control	Total infection averted	Total cost ($)	J
Strategy 2	$u_1=0, u_2=0, u_3\neq 0, u_4=0, u_5\neq 0$	0.0583	0.3716	3298.2
Strategy 4	$u_1=0, u_2=0, u_3\neq 0, u_4\neq 0, u_5\neq 0$	0.1457	2.5303	3756.3

The ICER for strategy 2 and 4 will become,

$$\text{ICER } 2=\frac{0.3716}{0.0583}=6.373927959 \quad , \quad \text{ICER } 4=\frac{2.5303-0.3716}{0.1457-0.0583}=24.69908467$$

The ICER (2) and ICER (4) shows a cost saving of $\$6.373927959$ for strategy 2 over strategy 4. That means, strategy 4 is more expensive and less effective than strategy 2. Therefore, strategy 4 is removed from the set of alternatives so that it does not consume more resources.

Table 5.2(iii): Strategies 2 and 5 are compared.

Strategy	Control	Total infection averted	Total cost ($)	J
Strategy 2	$u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 = 0, u_5 \neq 0$	0.0583	0.3716	3298.2
Strategy 5	$u_2 \neq 0, u_4 \neq 0, u_1 = u_3 = u_5 = 0$	0.1655	3.7126	3927.5

ICER for strategy 2 and Strategy 5 will be calculated as

$$\text{ICER 2} = \frac{0.3716}{0.0583} = 6.373927959 \qquad \text{ICER 5} = \frac{3.7126 - 0.3716}{0.1655 - 0.0583} = 31.16604478$$

The ICER (2) and ICER (5) shows a cost saving of $\$6.373927959$ for strategy 2 over strategy 5. That is, strategy 5 is expensive, so it is excluded from the set of alternatives.

Table 5.2 (iv): Strategies 2 and 6 are compared.

Strategy	Control	Total infection averted	Total cost ($)	J
Strategy 2	$u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 = 0, u_5 \neq 0$	0.0583	0.3716	3298.2
Strategy 6	$u_1 \neq u_2 \neq u_3 \neq u_4 \neq u_5 \neq 0$	0.1697	6.7670	1248.9

Then $\text{ICER 2} = \frac{0.3716}{0.0583} = 6.373927959$ and $\text{ICER 6} = \frac{6.7670 - 0.3716}{0.1697 - 0.0583} = 57.40933573$

This shows a cost saving of $\$6.373927959$ for strategy 2 over strategy 6. Strategy 6 is more costly than strategy 2, therefore strategy 6 is excluded

Table 5.2 (v): Strategies 2 and 7 are compared.

Strategy	Control	Total infection averted	Total cost ($)	J
Strategy 2	$u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 = 0, u_5 \neq 0$	0.0583	0.3716	3298.2
Strategy 7	$u_2 \neq 0, u_1 = u_3 = u_4 = u_5 = 0$	0.1709	2.5798	4086.7

Then, $\text{ICER 2} = \frac{0.3716}{0.0583} = 6.373927959$, $\text{ICER 7} = \frac{2.5798 - 0.3716}{0.1709 - 0.0583} = 19.61101243$

It shows a cost saving of $\$6.373927959$ for strategy 2 over strategy 7. The ICER for strategy 2 indicates that the strategy 7 is more expensive.

Table 5.2 (vi): Strategies 2 and 8 are compared.

Strategy	Control	Total infection averted	Total cost ($)	J
Strategy 2	$u_1 = 0, u_2 = 0, u_3 \neq 0, u_4 = 0, u_5 \neq 0$	0.0583	0.3716	3298.2
Strategy 8	$u_1 \neq 0, u_2 \neq 0, u_3 = u_4 = u_5 = 0$	0.1813	6.0612	1417.0

$$\text{ICER } 2 = \frac{0.3716}{0.0583} = 6.373927959$$

$$\text{ICER } 8 = \frac{6.0612 - 0.3716}{0.1813 - 0.0583} = 46.25691057$$

From this shows a cost saving of $\$6.373927959$ for strategy 2 over strategy 8. That is, strategy 8 is more expensive and less effective than strategy 2.Therefore, strategy 8, the strongly dominated is left out from the set of alternatives so that it does not consume limited resources.

With this result, we therefore conclude that strategy 2 (combination of removing vector breeding areas (u_3) and control maturation rate from larvae to adult (u_5)) is the most cost-effective of all the strategies for dengue fever disease control considered.

CHAPTER SIX

RESULTS AND DISCUSSION

A mathematical model with treatment, temporary immunity, careful and careless human susceptible for the transmission dynamics of dengue fever disease was presented. We present a $S_{h_{1,2}}ITRS_{h_1}$ (careful or careless susceptible, infected, treated, recovery, careful susceptible) and $A_mS_mI_m$ (aquatic, susceptible, infected) epidemic model to describe the interaction between human and dengue fever mosquito populations. In order to assess the transmission of Dengue fever disease, the susceptible population is divided into two, namely, careful and careless human susceptible population.

The model presents four possible equilibria: two disease-free and two endemic equilibrium points. The results show that the disease-free equilibrium point is locally and globally asymptotically stable if the reproduction number is less than unity. Endemic equilibrium point is locally and globally asymptotically stable under certain conditions using additive compound matrix and Lyapunov method respectively.

Modelling was examined with a special application to modelling dengue fever disease data in Tanzania. In particular, maximum likelihood estimator was explored and applied to the data from January 2010 to April 2015. The best fitting model based on how well the model captures the data. From the results, it is observed that the forecasted data agree very closely to the actual data.

Moreover statistical test is carried out in order to establish the relationship of forecasted and observed data. A statistical test showed significant correlation between predicted and observed number of cases in 2010-2015 (R =0.921, R square=0.848). Furthermore the correlation was significant. $\text{Predicted data} = 6.195 + 0.870 \times \text{Observed data}$, meaning that if observed data is increases by one, the model predict that, the predicted data increases by 0.870, so model was found to be valid since the statistic assumption is satisfied.

Sensitivity analysis of the model is implemented in order to investigate the sensitivity of certain key parameters of dengue fever disease with treatment, Careful and Careless Susceptibles on the transmission of Dengue fever Disease. Sensitivity analysis revealed that the most positive sensitive parameter is maturation rate from larvae to adult (per day). Simulation shows that when maturation rate from larvae to adult (per day) increases, the number of infected individuals increases while careful and careless susceptible decrease. This indicates that on the reduction of maturation rate from larvae to adult (per day), it is possible to maintain the effective reproduction number below unity and the disease can be eradicated from the community.

The model with treatment, temporary immunity, Careful and Careless Susceptibles was extended to include campaign of educating Careless human susceptible u_1, control vector-human contact u_2, reducing vector breeding areas u_3, insecticide application u_4 and control maturation rate from larvae to adult u_5. The results show that Treatment and the controls on the transmission of dengue fever disease will have a positive effect on decreasing the growth rate of dengue fever disease.

Furthermore optimal control analysis for dengue fever model was performed using Pontryagin's maximum principle. Conditions for optimal control of the disease were derived and analysed with an effective use of campaign to educate the careless human susceptible u_1, control vector-human contact u_2, removing vector breeding areas u_3, insecticides application u_4 and control maturation rate from larvae to adult u_5

The results suggest that the effective removing vector breeding areas (u_3) and control maturation rate from larvae to adult (u_5) strategy have a significant impact in reducing the dengue fever disease. From the cost-effectiveness analysis, the results suggest that combination of removing vector breeding areas (u_3) and control maturation rate from larvae to adult (u_5) are the most cost-effective of all the strategies for dengue fever disease control considered.

CHAPTER SEVEN

CONCLUSION AND RECOMMENDATIONS

We have presented mathematical model with treatment, temporary immunity, careful and careless human susceptible for the transmission dynamics of dengue fever disease and modified model incorporated with five controls that is campaign to educate the careless human susceptible u_1, control vector-human contact u_2, removing vector breeding areas u_3, insecticides application u_4 and control maturation rate from larvae to adult u_5. Optimal control analysis for dengue fever model was performed using Pontryagin's maximum principle to find which strategy minimizes the disease and cost. The results suggest that the effective removing vector breeding areas (u_3) and control maturation rate from larvae to adult (u_5) strategy have a significant impact in reducing the dengue fever disease. From the cost-effectiveness analysis, the results suggest that combination of removing vector breeding areas (u_3) and control maturation rate from larvae to adult (u_5) are the most cost-effective of all the strategies for dengue fever disease control considered.

Dengue fever eradication is currently a challenge in Tanzania and Sub-Saharan Africa, so there is a need of strengthening the control strategies, especially developing for effective treatment and vaccines which are not yet available. The incidence rate of dengue fever is so current in Tanzania and it seems to be increasing, thus due to the result of this work the following are recommended:-

(i) There is a need strengthening the control strategy especially for educating the society the prevalence of the disease and how to avoid getting the disease as at the moment there is no antiviral agents exist for dengue fever disease.

(ii) Patients should be advised to stay well hydrated and to avoid aspirin. For those who develop sever dengue, close observation and frequent monitoring in an intensive care unit setting may be required.

7.1 Future work

It is known that dengue fever disease have continued to spread worldwide and in Tanzania. There is need to have comprehensive researches aimed to explore possible new control strategies of the disease as well as assessing the impact of the existing control strategies. Based on the model of this study, it is proposed that future work should consider the following:

i. Investigate the impact of the drug resistance for those using treatment on the transmission of dengue fever infection.

ii. Investigate impact of pesticide application

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